Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

This paper is also taken for the relevant examination for the Associateship.

M2S1

Probability & Statistics II

Date: Tuesday, 13th May 2008 Time: 2 pm – 4 pm

Answer all questions. Each question carries equal weight.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Formula sheets are included on pages 4 & 5.

1. (a) An exam paper has an equal number of questions set by each of two examiners A and B. Each question is either easy or hard, and your probability of obtaining full marks on an easy question is $\frac{2}{3}$, whereas your probability of obtaining full marks on a hard question is $\frac{1}{4}$, irrespective of which examiner set the question. Examiner A sets questions which are easy with probability $\frac{1}{3}$ and hard with probability $\frac{2}{3}$. Examiner B sets only hard questions. You choose a question at random and obtain full marks for your answer. What is the probability that the question was set by examiner A?

(b) Let the random variables X and Y have joint probability density function

$$f_{X,Y}(x,y) = 2\exp(-x-y),\ 0 < x < y < \infty.$$

Are X and Y independent? Find their marginal probability density functions and their covariance.

[You may use the result that $2\int_0^\infty x^n \exp(-2x)dx = n!/2^n, n = 1, 2, \ldots$.]

(c) Let $X_1, \ldots, X_n$ $(n \geq 3)$ be independent, identically distributed $N(0,1)$ random variables.

(i) What is the distribution of $2X_1 - 3X_2$?

(ii) What is the distribution of $X_1^2 + X_2^2$?

(iii) What is the distribution of $\sqrt{2}X_1/\sqrt{X_2^2 + X_3^2}$?

(iv) What is the *joint* distribution of $\overline{X} = n^{-1}\sum_{i=1}^n X_i$ and $S^2 = \sum_{i=1}^n (X_i - \overline{X})^2$?

2. Let $X_1, \ldots, X_n$ be independent, identically distributed with common probability density function of the form

$$f(x;\mu) = \frac{1}{\mu}\exp\left(-\frac{x}{\mu}\right),\ x > 0,$$

where $\mu > 0$.

(i) What is the distribution of $\sum_{i=1}^n X_i$? What is the distribution of $2\sum_{i=1}^n X_i/\mu$?

Now let $Y_1, \ldots, Y_m$ be a further set of independent random variables from the density $f(x;\mu)$, independent of $X_1, \ldots, X_n$, and define $\overline{X} = n^{-1}\sum_{i=1}^n X_i$ and $\overline{Y} = m^{-1}\sum_{j=1}^m Y_j$.

(ii) What is the distribution of $\sum_{i=1}^n X_i / \sum_{j=1}^m Y_j$?

Let $T_p = p\overline{X} + (1-p)\overline{Y}$, where p is a fixed constant with $0 < p < 1$.

(iii) Find the mean and variance of T_p and show that, for all $\epsilon > 0$ and $0 < p < 1$,

$$P(|T_p - \mu| > \epsilon) \to 0,\ \text{as } m, n \to \infty.$$

Find the value of p that minimises the variance of T_p.

3. (a) State carefully the central limit theorem for identically distributed random variables with finite mean μ and finite variance σ^2.

(b) Suppose that $X_1, X_2, \dots$ are independent, identically distributed random variables each uniformly distributed over the interval $(0, 1)$.

Let $Y = \log(X_1)$.

(i) Find the probability density function of Y, and calculate the mean and variance of Y.

(ii) Suppose $0 < a < b$. Show that

$$P\{(X_1 X_2 \dots X_n)^{n^{-1/2}} e^{n^{1/2}} \in [a, b]\}$$

tends to a limit as $n \to \infty$ and find an expression for it.

(c) Let X have zero mean and variance σ^2. By noting that $P(X \geq t) \equiv P(X + \sigma^2/t \geq t + \sigma^2/t)$, show that

$$P(X \geq t) \leq \frac{\sigma^2}{\sigma^2 + t^2}, \quad \text{for } t > 0.$$

4. Let the random variable X have probability density function

$$f(x; \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \ x > 0, \ \text{where} \ \ \mu > 0, \ \lambda > 0.$$

(i) Verify that the moment generating function of X is of the form

$$M_X(t) = \exp\left\{\frac{\lambda}{\mu}\left(1 - \sqrt{1 - 2t\mu^2/\lambda}\right)\right\},$$

and hence, or otherwise, find the mean of X.

(ii) Identify the distribution of Y, where

$$Y = \frac{\lambda(X-\mu)^2}{\mu^2 X}.$$

(iii) Let $X_1, \dots, X_n$ be independent, identically distributed, with common density function $f(x; \mu, \lambda)$. Find the form of the maximum likelihood estimators $\widehat{\mu}, \widehat{\lambda}$ of μ, λ. What is the distribution of $\widehat{\mu}$?

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu + \sigma \mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2 \mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS							
	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma(1+2/\alpha)-\Gamma(1+1/\alpha)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	