Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2006

This paper is also taken for the relevant examination for the Associateship.

M2S1

PROBABILITY AND STATISTICS II

Date: Friday, 12th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used. A Formula Sheet is provided on pages 7-8.

1. (a) Discrete random variable X has range $\mathbb{X} \equiv \{0, 1, 2, \ldots\}$ and cumulative distribution function (cdf) F_X specified by

$$F_X(x) = \frac{3^{x+1} - k^{x+1}}{3^{x+1}} \qquad x = 0, 1, 2, \ldots$$

and $F_X(x) = 0$ for all other integers x, with the usual right continuous step function behaviour elsewhere, where k is a constant.

By noting that F_X can be written

$$F_X(x) = 1 - \theta^{x+1} \qquad x = 0, 1, 2, ...$$

for some θ, and thus is a modified version of a standard distribution, find

(i) the range of values of k that yield a proper probability distribution for X.

(ii) the probability mass function (pmf) of X, denoted f_X.

(iii) the moment generating function of X, $M_X(t)$.

(b) Suppose that X_1 and X_2 are independent random variables that have the same probability distribution as X from part (a). Let $Y = X_1 + X_2$, and $Z = \max\{X_1, X_2\}$.

Find

(i) the expectation of Y,

(ii) the mgf of Y,

(iii) the cdf of Z.

2. (a) Suppose that U is a continuous random variable, and $U \sim Uniform(0, 1)$. Let random variable X be defined in terms of U by

$$X = \sin(\pi U/2).$$

Find

(i) the probability density function (pdf) of X, denoted f_X,

(ii) the expectation $E_{f_X}[X]$,

(iii) the expected area of the (random) triangle with corners

$$(0, 0), (U, U/2), (U, -U/2).$$

(b) The joinf pdf of continuous random variables Y and Z is specified via the conditional distribution of Y given $Z = z$, and the marginal distribution for Z. Specifically,

$$\begin{aligned} Y|Z = z &\sim Uniform(0, \sqrt{z}) \\ Z &\sim Gamma(3/2, \lambda) \end{aligned}$$

for parameter $\lambda > 0$.

Find the marginal pdf for Y, denoted f_Y.

Hint: consider the support of the conditional pdf $f_{Y|Z}(y|z)$ carefully.

3. (a) Suppose that X_1 and X_2 are independent and identically distributed continuous random variables with cumulative distribution function

$$F_X(x) = \frac{x}{1+x} \qquad x > 0$$

with $F_X(x) = 0$ for $x \leq 0$.

Show that

$$P[X_1 X_2 < 1] = P[X_1 < 1].$$

(b) Suppose that Z_1 and Z_2 are independent random variables, where $Z_1 \sim Normal(0,1)$ and Z_2 is the absolute value of a $Normal(0,1)$ random variable, so that

$$f_{Z_2}(z) = \sqrt{\frac{2}{\pi}} \exp\{-z^2/2\} \qquad z > 0$$

and zero otherwise . Let

$$Y_1 = \frac{Z_1}{\sqrt{Z_1^2 + Z_2^2}} \qquad Y_2 = \sqrt{Z_1^2 + Z_2^2}.$$

Find the marginal probability density function of Y_1.

Are Y_1 and Y_2 independent ? Justify your answer.

4. (a) Discrete random variables X_1 and X_2 have joint probability mass function specified by the following table:

		X_1 0	1	2
	0	p	$2p$	0
X_2	1	p	p	p
	2	0	0	$4p$

(i) Find the value of the constant p.

(ii) Find the covariance between X_1 and X_2.

(b) Suppose that Z_1, Z_2 and Z_3 are independent random variables each having a $Normal(0,1)$ distribution. Suppose that Y_1, Y_2 and Y_3 are defined in terms of Z_1, Z_2 and Z_3 by the equations

$$\begin{aligned} Y_1 &= Z_1 + Z_2 \\ Y_2 &= 2Z_1 + Z_3 \\ Y_3 &= Z_1 + Z_2 - 2Z_3. \end{aligned}$$

Find the variance/covariance matrix for the random variables (Y_1, Y_2, Y_3).

Are Y_2 and Y_3 independent random variables ? Justify your answer.

5. (a) Suppose $X_1, \ldots, X_n, \ldots$ is a sequence of random variables with the cumulative distribution function of X_n defined by

$$F_{X_n}(x) = \left(\frac{1}{1+e^{-x}}\right)^n \qquad x \in \mathbb{R}.$$

Find the limiting distributions as $n \longrightarrow \infty$ (if they exist) of the random variables

(i) X_n,

(ii) $U_n = X_n - \log n$.

Using the result in (ii), find an approximation to the probability

$$P[X_n > k]$$

for large n.

(b) In a dice rolling game, a fair die (with all six scores having equal probability) is rolled repeatedly and independently under identical conditions. On each roll, the player wins six points if the score is a 6, loses one point if the score is either 2,3,4 or 5, and loses two points if the score is 1.

Let T_n denote the points total obtained after n rolls of the die. The player begins the game with a points total equal to zero, that is $T_0 = 0$.

(i) Find the expectation and variance of the points total after 100 rolls of the die.

(ii) Find an approximation to the distribution of the points total after n rolls, for large n.

(iii) Describe the behaviour of the sample mean points total, $M_n = T_n/n$, as $n \longrightarrow \infty$.

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu + \sigma\mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2\mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma(1+2/\alpha)-\Gamma(1+1/\alpha)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\left\{1+\dfrac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	