1. (a) Suppose that discrete random variable X has range $\mathbb{X} \equiv \{1, 2, 3, ...\}$ and probability mass function (pmf) f_X given by

$$f_X(x) = \frac{-1}{\log(1-\phi)} \frac{\phi^x}{x} \qquad x = 1, 2, 3, ...$$

and zero otherwise, for parameter ϕ, where $0 < \phi < 1$.

(i) Prove that f_X is a valid pmf.

(ii) Find the expectation of X.

(iii) Find the moment generating function (mgf) for $X, M_X(t)$.

(b) Discrete random variable Y has a *mixture distribution,* that is, its pmf f_Y takes the form

$$f_Y(y) = \alpha f_1(y) + (1-\alpha) f_2(y)$$

for $y \in \mathbb{Y}$, where $0 < \alpha < 1$, and f_1 and f_2 are also discrete pmfs.

Let $M_1(t)$ and $M_2(t)$ denote the mgfs of, and μ_1 and μ_2 the expectations with respect to the pmfs f_1 and f_2, respectively. Prove from the definition of expectation that

$$E_{f_Y}[Y] = \alpha\mu_1 + (1-\alpha)\mu_2$$

and find the form of the mgf of Y in terms of α, M_1 and M_2.

2. (a) (i) Suppose that random variable X has a *Gamma*$(\alpha, 1)$ distribution, for parameter $\alpha > 0$. Find the probability density function (pdf) of random variable Y defined by

$$Y = \frac{1}{X}.$$

(ii) Suppose that continuous random variable U has a cumulative distribution function (cdf), F_U, given by

$$F_U(u) = \exp\{-\exp\{-u\}\} \qquad u \in \mathbb{R}.$$

Find the pdf of random variable V defined by

$$V = U^2.$$

(b) Suppose that X and Y are positive, independent continuous random variables with cdfs F_X and F_Y and pdfs f_X and f_Y respectively.

Show that

$$P[X < Y] = \int\int_A f_X(x) f_Y(y) dx dy$$

for a suitably defined set A.

Deduce that

$$P[X < Y] = \int_0^1 F_X\left(F_Y^{-1}(t)\right) dt,$$

where F_Y^{-1} is the inverse function for the 1-1 function F_Y.

Hint: integrate dx, then change variables in the remaining dy integral.

3. (a) Suppose that Z_1 and Z_2 are independent random variables each having an *Exponential* (1) distribution.

   (i) Find the joint pdf of random variables Y_1 and Y_2 defined by

$$Y_1 = \frac{Z_1}{Z_1 + Z_2}, \qquad Y_2 = Z_1 + Z_2.$$

   (ii) Find the conditional pdf of Y_1, given that $Y_2 = 1$.

(b) For random variables X and Y, using the following representation of the joint probability model:

$$f_{X,Y}(x, y) = f_{X|Y}(x|y)\, f_Y(y),$$

compute the marginal probability distribution of X when,

   (i)

$$X|Y = y \sim \textit{Binomial}(n, y)$$

   for positive integer n, and the continuous random variable Y has a standard uniform marginal distribution.

   (ii)

$$X|Y = y \sim \textit{Exponential}(y)$$

   and $Y \sim$ *Exponential*(β) for parameter $\beta > 0$.

State the range of the X variable in each case.

4. (a) Compute the correlation between continuous random variables X and Y where

$$X \sim \textit{Normal}\,(0,1)$$

and $Y = X^2$.

Are X and Y independent ? Justify your answer.

Hints:

(i) *Use properties of expectations and the following results for the standard normal distribution – if $Z \sim N(0,1)$, then*

$$E_{f_Z}[Z] = 0 \qquad E_{f_Z}\left[Z^2\right] = 1 \qquad E_{f_Z}\left[Z^3\right] = 0 \qquad E_{f_Z}\left[Z^4\right] = 3.$$

(ii) *If $Y = X^2$, the rules of expectation dictate that for a general function h,*

$$E_{f_{X,Y}}[h(X,Y)] \equiv E_{f_X}\left[h\left(X,X^2\right)\right].$$

(b) Suppose that X_1 and X_2 are independent standard normal random variables. Define random variables Y_1 and Y_2 by the multivariate linear transformation

$$Y = AX + b$$

where $X = (X_1, X_2)^\top$ and $Y = (Y_1, Y_2)^\top$ are the column vector random variables, A is the 2×2 matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

and $b = (1,2)^\top$ is a constant column vector.

Find

(i) The marginal distribution of Y_1.

(ii) The covariance and correlation between Y_1 and Y_2.

You may quote without proof general properties of the univariate and multivariate normal distribution.

5. (a) Suppose that continuous random variables X_n, for $n = 1, 2, 3...$, each with range $\mathbb{X} \equiv (0, \infty)$, have cdfs given by

$$F_{X_n}(x) = \left(\frac{x^2}{1+x^2}\right)^n \qquad x > 0$$

and zero otherwise. Show that, in the limit as $n \to \infty$,

(i) the limiting distribution of X_n does not exist,

(ii) the limiting distribution of Y_n defined by

$$Y_n = X_n/\sqrt{n}$$

does exist and is a continuous distribution on $\mathbb{X}$.

(b) In a simple model of the Foreign Exchange market, the exchange rate for the British Pound against the US Dollar is believed to change from day to day according to the following simple probability model; from one day to the next, the rate **increases** by a constant multiplicative factor $a > 1$ with probability $\frac{1}{2}$ and **decreases** by a factor $1/a < 1$ with probability $\frac{1}{2}$. On the log scale, therefore, the log exchange rate on day n is a random variable Y_n, say, where

$$Y_n = \sum_{i=0}^{n} X_i$$

where, for $i = 1, 2, 3....$

$$X_i = \begin{cases} \log a & \text{with probability } \frac{1}{2} \\ -\log a & \text{with probability } \frac{1}{2} \end{cases}$$

and for $i = 0$, $X_0 = x_0$, a known constant.

(i) Using the Central Limit Theorem, find an approximation to the distribution of Y_n for large n.

(ii) Show that, for any n,

$$P\left[|Y_n - x_0| \geq 2\sigma_n\right] \leq \frac{1}{4}$$

where

$$\sigma_n^2 = n\left(\log a\right)^2.$$

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (given on Page 8), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathrm{E}_{f_Y}[Y] = \mu + \sigma\mathrm{E}_{f_X}[X] \qquad \mathrm{Var}_{f_Y}[Y] = \sigma^2\mathrm{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	PDF f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma\left(1+\alpha^{-1}\right)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma\left(1+2\alpha^{-1}\right)-\Gamma\left(1+\alpha^{-1}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$\exp\left\{\mu t+\dfrac{\sigma^2t^2}{2}\right\}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{\Gamma\left(\dfrac{\nu+1}{2}\right)}{\Gamma\left(\dfrac{\nu}{2}\right)\sqrt{\pi\nu}\left\{1+\dfrac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	