Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M2S1 PROBABILITY AND STATISTICS II

Date: Tuesday, 18th May 2004 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

Formula sheets are included on pages 7 & 8

1. **(a)** Suppose that $U \sim Uniform\,(0,1)$, so that

$$F_U(u) = u \qquad 0 < u < 1$$

with $F_U(u) = 0$ for $u \leq 0$ and $F_U(u) = 1$ for $u \geq 1$.

(i) Find the probability density function (pdf), f_X, of random variable X defined by

$$X = \ln\left(\frac{U}{1-U}\right)$$

and show that, for $x \in \mathbb{R}$,

$$f_X\,(x) = f_X\,(-x)$$

(ii) Find the expectation of X.

(iii) By using the substitution

$$v = \frac{1}{1+e^x}$$

in the integral, or otherwise, show that the moment generating function (mgf) of X, M_X, is given by the expression

$$M_X\,(t) = \int_0^1 \left(\frac{1-v}{v}\right)^t dv \qquad -1 < t < 1.$$

(b) Suppose that the continuous random variable Y has pdf f_Y and cdf F_Y. Let W be defined by

$$W = F_Y\,(Y)\,.$$

Show that, for $r = 1, 2, 3, ...$

$$E_{f_W}\,[W^r] = \frac{1}{r+1}$$

2. **(a)** Suppose that random variable Z has a standard Normal distribution.

(i) Compute the first four moments for Z,

$$E_{f_Z}\left[Z^r\right]$$

for $r = 1, 2, 3, 4$.

(ii) Suppose that $X \sim N(\mu, 1)$. Find the expectation and variance of random variable Y given by

$$Y = X^2.$$

(b) Suppose that the joint pdf of two variables U and V, $f_{U,V}$, is specified as follows:

$$f_{U,V}(u, v) = f_{U|V}(u|v)\, f_V(v)$$

where

$$U|V = v \sim Gamma\left(\frac{\alpha}{2} + v, \frac{1}{2}\right) \qquad V \sim Poisson(\lambda)$$

(i) Using the law of iterated expectation, find the marginal expectation of U, $E_{f_U}[U]$

(ii) Find the mgf of U.

3. **(a)** Suppose that Z_1 and Z_2 are independent random variables, each having a standard Normal distribution.

(i) Find the marginal pdf of Y_1, the random variable defined by

$$Y_1 = \frac{Z_1}{Z_2}.$$

(ii) Find the expectation of Y_1.

(b) Suppose that V_1 and V_2 are independent random variables, where

$$V_1 \sim Gamma\,(2, \beta) \qquad\qquad V_2 \sim Gamma\,(4, \beta)$$

for parameter $\beta > 0$. Let the set A_w be defined for $0 < w < 1$ by

$$A_w \equiv \{(v_1, v_2) : (1 - w)\, v_1 \le w v_2,\ v_1 > 0,\ v_2 > 0\}\,.$$

(i) Show that

$$P\,[(V_1, V_2) \in A_w] = \int_0^\infty F_{V_1}\left(\frac{w v_2}{1 - w}\right) f_{V_2}\,(v_2)\, dv_2$$

where F_{V_1} is the cdf of V_1.

(ii) By computing F_{V_1}, find an explicit expression for $P\,[(V_1, V_2) \in A_w]$ as a function of w.

Recall the Gamma function recursion formula; for $\alpha > 0$

$$\Gamma\,(\alpha) = (\alpha - 1)\,\Gamma\,(\alpha - 1)\,.$$

4. **(a)** The joint probability mass function (pmf) for discrete variables X and Y, each taking values in the set $\{1, 2\}$ is given by the following probability table

		X = 1	X = 2
Y	1	$\frac{1}{8}$	$\frac{1}{2}$
	2	$\frac{1}{4}$	$\frac{1}{8}$

Find the correlation between X and Y.

(b) **(i)** For X and Y as given in part **(a)**, find the variance of discrete random variable

$$T = X - Y$$

and show that

$$Var_{f_T}[T] \neq Var_{f_X}[X] + Var_{f_Y}[Y].$$

(ii) Prove from first principles that, in general, for two negatively correlated discrete random variables V_1 and V_2,

$$Var_{f_{V_1,V_2}}[V_1 - V_2] > Var_{f_{V_1}}[V_1] + Var_{f_{V_2}}[V_2]$$

5. **(a)** **(i)** Suppose that random variable X has a Poisson distribution with parameter λ. Consider the standardized random variable, Z_λ, defined by

$$Z_\lambda = \frac{X - \lambda}{\sqrt{\lambda}}.$$

Prove that, as $\lambda \to \infty$,

$$Z_\lambda \xrightarrow{d} Z \sim N(0,1).$$

(ii) Suppose that $X_1, ...X_n \sim Poisson(\lambda_X)$ and $Y_1, ...Y_n \sim Poisson(\lambda_Y)$, with all variables mutually independent. Find μ such that the random variable M defined by

$$M = \overline{X} + \overline{Y}$$

satisfies

$$M \xrightarrow{p} \mu$$

where

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

are the sample mean random variables for the two samples respectively.
[*State without proof any theorems that you use in giving the result*]

(b) Suppose that $X_1, ..., X_n \sim Exponential(\lambda)$. The cdf of the random variable $T_n = \max\{X_1, ..., X_n\}$ is given by

$$F_{T_n}(t) = \{F_X(t)\}^n.$$

where F_X is the cdf of $X_1, ..., X_n$.

(i) Find $F_{T_n}(t)$ explicitly.

(ii) Discuss the form of the limiting distribution of T_n as $n \longrightarrow \infty$.

(iii) Find the form of the limiting distribution of random variable U_n, defined by

$$U_n = \lambda T_n - \log n$$

as $n \longrightarrow \infty$.

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\frac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\frac{1}{\theta}$	$\frac{(1-\theta)}{\theta^2}$	$\frac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\frac{n}{\theta}$	$\frac{n(1-\theta)}{\theta^2}$	$\left(\frac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\frac{n(1-\theta)}{\theta}$	$\frac{n(1-\theta)}{\theta^2}$	$\left(\frac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (given on Page 8), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathrm{E}_{f_Y}[Y] = \mu + \sigma\mathrm{E}_{f_X}[X] \qquad \mathrm{Var}_{f_Y}[Y] = \sigma^2\mathrm{Var}_{f_X}[X]$$

					$\frac{\nu}{}$ (if $\alpha > 2$)
$Beta(\alpha, \beta)$	$(0,1)$	$\alpha, \beta \in \mathbb{R}^+$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$