UNIVERSITY OF LONDON

Course:	M2PM4
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Date:	May 8, 2008

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

M2PM4

Rings and fields

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Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

This paper is also taken for the relevant examination for the Associateship.

M2PM4

Rings and fields

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Give the definitions of an ideal of a ring, of a principal ideal, and of a principal ideal domain.

(b) Working from the definition prove that the polynomial ring $\mathbb{Q}[x]$ is a principal ideal domain.

(c) Prove that if I is a maximal ideal of a ring R with 1, then R/I is a field.

(d) Is $\mathbb{Z}[\sqrt{-11}]$ a Euclidean domain? (Justify your answer. You can use any results from lectures if you state them explicitly.)

2. In this question you are asked to justify your answers. You can use any results from lectures if you state them explicitly.

(a) Find all prime numbers p such that $x^3 - 24x - p$ is irreducible in $\mathbb{Q}[x]$.

(b) Determine which of the following ideals are maximal:

the principal ideal generated by $x^2 + 1$ in the ring $\mathbb{R}[x]$,

the principal ideal generated by 5 in the ring $\mathbb{Z}[\sqrt{-5}]$,

the principal ideal generated by 6 in the ring $\mathbb{Z}/24$.

(c) Find a maximal ideal I in $\mathbb{Z}[\sqrt{-3}]$ such that the characteristic of the field $\mathbb{Z}[\sqrt{-3}]/I$ is 7.

(d) Prove that if I is an ideal of $\mathbb{Z}[x]$ such that the factor ring $\mathbb{Z}[x]/I$ is finite and non-zero, then I is not a principal ideal.

3. (a) Let $F \subset K \subset E$ be finite extensions of fields. Prove that $[E : F] = [E : K][K : F]$.

(b) Do there exist finite fields $F \subset K$ such that $|F| = 8$ and $|K| = 16$? If you think that the answer is 'yes', you need to explain how these fields can be constructed. If you think that the answer is 'no', you need to prove that such fields do not exist.

(c) Let $F \subset K$ be fields such that $[K : F] = 7$. Prove that for any $\alpha \in K$, $\alpha \notin F$, we have $K = F(\alpha)$.

(d) What is the degree of the minimal polynomial of $\sqrt{7} + \sqrt{-2}$ over $\mathbb{Q}$? Justify your answer.

4. (a) Explain why a regular polygon with 7 sides cannot be constructed using only a ruler and a compass.

(b) Explain why a regular polygon with 15 sides can be constructed using only a ruler and a compass.

In parts (a) and (b) a few sentences will suffice; you are not asked to give full details of your argument.

(c) Let F be the field of four elements $(\mathbb{Z}/2)(\omega)$, where $\omega^2 + \omega + 1 = 0$. Is the polynomial $x^3 + x + \omega$ irreducible in $F[x]$? Justify your answer.