M2PM3 SOLUTIONS 8, 26.3.2009

Q1. With $z = e^{i\theta}$, $\cos\theta = \frac{1}{2}(z + z^{-1})$, $\sin\theta = \frac{1}{2i}(z - z^{-1})$, $dz/iz = d\theta$. So with Γ the unit circle, the integral is

$$I = \int_\Gamma \frac{\frac{1}{(-4)}(z - z^{-1})^2}{a + \frac{b}{2}(z + z^{-1})} \frac{dz}{iz} = \frac{i}{2b} \int_\Gamma \frac{(z^2 - 1)^2 dz}{z^2(z^2 + 2az/b + 1)}.$$

The quadratic in the denominator has roots $\alpha = (-a + \sqrt{a^2 - b^2})/b$, $\beta = (-a - \sqrt{.})/b$. The product of the roots is 1, so $|\alpha| < 1$, $|b| > 1$. So the integrand $f(z) = (z^2 - 1)^2/[z^2(z - \alpha)(z - \beta)]$ is holomorphic inside Γ except for a double pole at 0 and a simple pole at α. By the Cover-Up Rule, as $1/\alpha = \beta$,

$$Res_\alpha f = (\alpha^2 - 1)^2/[\alpha^2(\alpha - \beta)] = (\alpha - 1/\alpha)^2/(\alpha - \beta) = (\alpha - \beta)^2/(\alpha - \beta)$$

$$= \alpha - \beta = 2\sqrt{a^2 - b^2}/b.$$

Near 0,

$$f(z) = \frac{1 - 2z^2 + ..}{z^2(1 + 2az/b + z^2} = \frac{1}{z^2}(1 - 2az/b + O(z^2))(1 + O(z^2)) = \frac{1}{z^2}[1 - 2az/b + O(z^2)].$$

So picking out the coefficient of $1/z$ (the residue, by definition), $Res_0 f = -2a/b$. So by Cauchy's Residue Theorem,

$$I = \frac{i}{2b}.2\pi i.\Big(-\frac{2a}{b} + \frac{2\sqrt{a^2 - b^2}}{b}\Big) = \frac{2\pi}{b}[a - \sqrt{a^2 - b^2}].$$

Q2. As the integrand is even, it suffices to show

$$I := \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi}{\sqrt{2}a^3}.$$

Use $f(z) = 1/(z^4 + a^4)$ round a semicircular contour Γ in the upper half-plane with base $[-R, R]$. As $R \to \infty$, the contribution from the semicircle is $O(1/R^3) \to 0$, and the contribution from the base tends to I. The integrand has poles where $z^4 = -a^4 = e^{i\pi}a^4 = e^{i(2n+1)\pi}a^4$, $z = ae^{i\pi/4}$, $ae^{3i\pi/4}$, $ae^{5i\pi/4}$, $ae^{7i\pi/4}$. Only the first two of these lie in the upper half-plane. So f is holomorphic inside Γ except for simple poles at $ae^{i\pi/4}$, $ae^{3i\pi/4}$. Let b stand for any of the poles. As the residue of f at b is the coefficient of $1/(z - b)$ in the Laurent expansion of f at b, we can evaluate this by multiplying f by $z - b$ and letting $z \to b$:

$$Res_b f = \lim_{z \to b} (z - b)/(z^4 - b^4) = 1/4b^3 = b/4b^4 = -b/4a^4,$$

by L'Hospital's Rule (quicker here than the Cover-Up Rule). So

$$Res_{e^{i\pi/4}} f = -\frac{e^{i\pi/4}}{a^4}, \qquad Res_{e^{3i\pi/4}} f = -\frac{e^{3i\pi/4}}{a^4}.$$

So by Cauchy's Residue Theorem, $I = 2\pi i.(-)(ae^{i\pi/4} + ae^{3i\pi/4})/4a^4$, or

$$I = -\frac{i\pi}{2a^3}(e^{i\pi/4} - e^{-i\pi/4}) = -\frac{i\pi}{2a^3}.2i\sin(\pi/4) = \frac{\pi}{\sqrt{2}a^3}.$$

Q3. Put $f(z) = (\pi\cot\pi z)/(1+z+z^2)$. Since $z^3 - 1 = (z-1)(z^2+z+1)$, the roots of z^2+z+1 are $e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$ and $e^{4\pi i/3} = -1/2 - i\sqrt{3}/2$, the complex cube roots of unity other than 1. Integrating f round the square contour Γ_n with vertices $(n+1/2)(\pm 1 \pm i)$ gives

$$\int_{\Gamma_n} f = 2\pi i\Big(\sum_{k=-n}^{n} \frac{1}{1+k+k^2} + Res_{e^{2\pi i/3}} f + Res_{e^{4\pi i/3}} f\Big).$$

But by the Estimation Lemma (or ML Inequality),

$$\int_{\Gamma_n} f = O(1/n^2).O(n) = O(1/n) \to 0 \qquad (n \to \infty).$$

Combining,

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n+n^2} = -\Big(Res_{e^{2\pi i/3}} + Res_{e^{4\pi i/3}}\Big)\frac{\pi\cot\pi z}{(z-e^{2\pi i/3})(z-e^{4\pi i/3})}.$$

By the Cover-Up Rule, the RHS is

$$-\frac{\pi\cot(\pi e^{2\pi i/3})}{i\sqrt{3}} + \frac{\pi\cot(\pi e^{4\pi i/3})}{i\sqrt{3}} = \frac{i\pi}{\sqrt{3}}[(\cot(-\frac{\pi}{2} + \frac{i\pi\sqrt{3}}{2}) - \cot(-\frac{\pi}{2} - \frac{i\pi\sqrt{3}}{2})].$$

Since $\tan(a + \pi/2) = -\cot a$, $\cot(a - \pi/2) = -\tan a$, the RHS is

$$\frac{i\pi}{\sqrt{3}}[\tan(-\frac{i\pi\sqrt{3}}{2}) - \tan(\frac{i\pi\sqrt{3}}{2})] = \frac{2i\pi}{\sqrt{3}}\tan(-\frac{i\pi\sqrt{3}}{2}).$$

As $i\tan i\theta = \tanh\theta$, this is $(2i\pi/\sqrt{3}).(-i).\tanh(\pi\sqrt{3}/2)$. So

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n+n^2} = \frac{2\pi}{\sqrt{3}}\tanh(\pi\sqrt{3}/2).$$

Q4. For $m > 0$, put $u := mx$. Since $dx/x = du/u$, this reduces the problem to the case $m = 1$, which gives $I = \pi/2$ (Lectures). For $m < 0$, we get $I = -\pi/2$, since the integrand is odd in m. For $m = 0$, we get 0 since the integrand is 0.

NHB