M2PM3 SOLUTIONS 7. 19.3.2009

Q1. Since $(\sin x)/x$ is bounded on $[0,\infty)$, there is no problem about the integral existing, and one can show that F is continuous on $[0,\infty)$; it is clearly decreasing. So $I = F(0) = F(0+)$, by continuity. Differentiating under the integral sign (which we may – given) gives

$$\begin{aligned}
F'(t) &= -\int_0^\infty e^{-xt}\sin x dx \\
&= \int_0^\infty e^{-xt} d\cos x \\
&= [e^{-xt}\cos x]_0^\infty - \int_0^\infty \cos x.(-t)e^{-xt}dx \\
&= -1 + t\int_0^\infty e^{-xt} d\sin x \\
&= -1 + t[e^{-xt}\sin x]_0^\infty - t\int_0^\infty \sin x.(-t)e^{-xt}dx \\
&= -1 + t^2\int_0^\infty e^{-xt}\sin x dx \\
&= -1 - t^2F'(t) :
\end{aligned}$$

$$(1+t^2)F'(t) = -1, \qquad F'(t) = -1/(1+t^2).$$

Integrating, $F(t) = -\tan^{-1} t + C$. But $F(t) \to 0$ as $t \to \infty$: $C = +\tan^{-1}\infty = \pi/2$. So $I = F(0+) = \pi/2$.

Q2. As $x^4+5x^2+6 = (x^2+3)(x^2+2)$, which has simple zeros at $\pm i\sqrt{2}$, $\pm i\sqrt{3}$, use $f(z) := z^2/(z^2+3)(z^2+2)$ and the contour Γ consisting of a large semicircle in the upper half-plane with base $[-R,R]$. Then f has simple poles inside Γ at $i\sqrt{3}$, $i\sqrt{2}$. By Jordan's Lemma, the integral round the semicircle tends to 0 as $R \to \infty$, while the integral along the base tends to the integral I we are to evaluate. So by Cauchy's Residue Theorem, $I = \sum Res f$, the sum being over the poles at $i\sqrt{3}$ and $i\sqrt{2}$ inside Γ. As both poles are simple, we can use the Cover-Up Rule:

$$Res_{i\sqrt{3}}f = (i\sqrt{3})^2/[(-3+2)(i\sqrt{3}+i\sqrt{3})] = (-3)/[-2i\sqrt{3}] = -i\sqrt{3}/2;$$

$$Res_{i\sqrt{2}}f = (i\sqrt{2})^2/[(-2+3)(i\sqrt{2}+i\sqrt{2})] = (-2)/[2i\sqrt{2}] = i\sqrt{2}/2.$$

So by Cauchy's Residue Theorem,

$$I = 2\pi i\sum Res f = 2\pi i.(-)i(\sqrt{3}-\sqrt{2})/2 = \pi(\sqrt{3}-\sqrt{2}).$$

Q3. Use $f(z) = (e^{ipz} - e^{iqz})/z^2$. This has a pole at 0 (apparently double, but actually single: the numerator has a simple zero at 0). We use a semicircular

contour, indented to avoid this pole – a contour Γ consisting of:
(i) Γ_1, the line segment $[-R,-r]$ (R large, $r>0$ small);
(ii) Γ_2, the semi-circle centre 0 radius r in the upper half-plane, clockwise (-ve sense);
(iii) $\Gamma_3 = [r,R]$;
(iv) Γ_4, the semi-circle centre 0 radius R in the upper half-plane, anticlockwise (+ve sense).
On Γ_4, $|e^{ipz}| = |e^{ip(x+iy)}| = e^{-py} \leq 1$, as $p \geq 0$ and $y \geq 0$ in the upper half-plane, and similarly $|e^{iqz}| \leq 1$. So $|f(z)| = O(1/R^2)$, and by (ML),
$\int_{\Gamma_4} f = O(1/R^2).\pi R = O(1/R) \to 0$ as $R \to \infty$.
As $R \to \infty, r \to 0$, $\int_{\Gamma_1} f + \int_{\Gamma_3} f \to I$, the required integral.
On Γ_2, $z = re^{i\theta}$,

$$f(z) = [(1+ipz-p^2z^2/2...)-(1+iqz-q^2z^2/2...)]/z^2 = [i(p-q)-\frac{1}{2}(p^2-q^2)z+...]/z,$$

$dz/z = ire^{i\theta}d\theta/re^{i\theta} = id\theta$, where θ goes from π to 0. So (changing the sign to interchange the limits of integration)

$$\int_{\Gamma_2} f = -\int_0^\pi [i(p-q)+O(r)](id\theta) \to \pi(p-q) \qquad (r \to 0).$$

Since $\int_\Gamma f = 0$ by Cauchy's Theorem, this gives $I + \pi(p-q) = 0$: $I = -\pi(p-q)$.

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