M2PM3 SOLUTIONS 6. 12.3.2009

Q1 (*Poisson kernel*).
(i) $(w-z)(\bar{w}-\bar{z}) = (w-z)\overline{(w-z)} = |w-z|^2$. Also

$$(w+z)(\bar{w}-\bar{z}) = w\bar{w}-w\bar{z}+z\bar{w}-z\bar{z} = |w|^2-|z|^2-((w\bar{z})-\overline{(w\bar{z})} = |w|^2-|z|^2-2\Im(w\bar{z}).$$

So multiplying top and bottom by $\bar{w}-\bar{z}$,

$$\frac{w+z}{w-z} = \frac{|w|^2-|z|^2-2\Im(w\bar{z})}{|w-z|^2},$$

and the result follows on taking real parts.
(ii) $|w-z|^2 = (w-z)(\bar{w}-\bar{z}) = w\bar{w}-w\bar{z}-\bar{w}z+z\bar{z} = |w|^2-[(w\bar{z})+\overline{(w\bar{z})}]+|z|^2$
$= R^2 - 2\Re(Re^{i\phi}.re^{-i\theta}) + r^2 = R^2 - 2Rr\cos(\theta-\phi)+r^2$.
(iii) Combining,

$$\Re\Big(\frac{w+z}{w-z}\Big) = \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}.$$

Q2 (*Poisson integral*).
$g(w)$ is holomorphic except where $w = R^2/\bar{z} = (R^2/r)e^{-i\theta}$, which is outside D as $r < R$. So as f is holomorphic in D, so is fg. So by Cauchy's Integral Formula ,

$$f(z)g(z) = \frac{1}{2\pi i}\int_{C(0,R)}\frac{f(w)g(w)}{w-z}dw.$$

But $g(z) = (R^2-r^2)/(R^2-z\bar{z}) = (R^2-r^2)/(R^2-|z|^2) = 1$ as $|z| = r$. So this gives

$$f(z) = \frac{R^2-r^2}{2\pi i}\int_{C(0,R)}\frac{f(w)}{(w-z)(R^2-w\bar{z})}dw.$$

The right is

$$\frac{R^2-r^2}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\phi})}{(Re^{i\phi}-re^{i\theta})(R^2-Rre^{i(\phi-\theta)})}.iRe^{i\phi}d\phi,$$

or

$$\frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(Re^{i\phi})}{(R-re^{i(\theta-\phi)})(R-re^{i(\phi-\theta)})}.d\phi.$$

So

$$f(re^{i\theta}) = \frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(Re^{i\phi})}{(R^2-2Rr\cos(\theta-\phi)+r^2)}.d\phi.$$

Note. We now know that harmonic functions u are exactly the real parts of holomorphic functions f. So taking real parts of $f = u+iv$:

$$u(re^{i\theta}) = \frac{R^2-r^2}{2\pi i}\int_0^{2\pi}\frac{u(Re^{i\phi})}{(R^2-2Rr\cos(\theta-\phi)+r^2)}.d\phi.$$

These are the *Poisson integral formulae*, giving a holomorphic or harmonic function inside a disc in terms of an integral involving its values on the boundary.

Q3 *Schwarz's Formula.*
(i)

$$\frac{d}{dz}\Big(\frac{w+z}{w-z}\Big) = \frac{d}{dz}\Big(1+\frac{2z}{w-z}\Big) = \frac{2[(w-z)-z(-1)]}{(w-z)^2} = \frac{2w}{(w-z)^2}.$$

For $|z| \le R' < R$, we can justify differentiating under the integral sign, which gives

$$f'(z) = \frac{1}{2\pi i}\int_{C(0,R)} \frac{2w}{(w-z)^2}.u(w)\frac{dw}{w}.$$

[Differentiate the RHS from first principles. As $|w| = R$ and $|z| \le R' < R$, $|w-z| \le R-R'$, so $1/(|w-z|^2 \le 1/(R-R')^2 < \infty$.] So f is holomorphic, as required.
(ii) If $w = Re^{i\phi}$, $dw = iRe^{i\phi}d\phi$, $dw/(iw) = d\phi$ real. So taking real parts of (i) gives

$$u(z) = \frac{1}{2\pi i}\int_{C(0,R)} \Re\Big(\frac{w+z}{w-z}\Big)u(w)\frac{dw}{w}$$

immediately. Then
(iii)

$$u(z) = \frac{R^2-r^2}{2\pi}\int_0^{2\pi} \frac{u(Re^{i\phi})}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi \qquad (z = re^{i\phi}).$$

(iv) This follows from (i).
(v) If $f = u+iv$, $-if = v-iu$. So by (ii), the values of $-if$ in the disc follow from the boundary values of v. So the values of f do also.

Q3 (*Liouville's theorem on* $\mathbf{C}^*$). As f is holomorphic for all $|z| \ge 1$ (including $+\infty$), $f(1/z)$ is holomorphic, and so continuous, in the closed unit disc $\bar{D} := \{z : |z| \le 1\}$. So as $\bar{D}$ is compact, $f(1/z)$ is bounded on $\bar{D}$: $|f(1/z)| \le M_1$, say, for $|z| \le 1$, or $|f(z)| \le M_1$ for $|z| \ge 1$. Similarly, as $f(z)$ is holomorphic, so continuous, in $\bar{D}$, f is bounded on $\bar{D}$: $|f(z)| \le M_2$, say, for $|z| \le 1$. So if $M := \max(M_1, M_2)$, $|f(z)| \le M$ for all z in $\mathbf{C}$: f is bounded. As f is also holomorphic in $\mathbf{C}$, so entire, f is constant, by Liouville's theorem.

So if f is entire and non-constant, f has a singularity at ∞.

Examples:

polynomials (non-constant – of degree ≥ 1);
exponentials (e^z, e^{z^2}, etc.);
trig functions ($\sin z$, $\cos z$), etc.

NHB