M2PM3 SOLUTIONS 3, 12.2.2009

Q1 *The Gamma function (real case).*
(i) Split $\int_0^\infty$ into $\int_0^1$ and $\int_1^\infty$. The second is convergent for all x, as the integrand decreases exponentially. For the first, the factor e^{-t} in the integrand is between e^{-1} and 1, so it suffices to consider $\int_0^1 t^{x-1}dt$. This is $[t^x/x]_0^1$ for $x > 0$, giving convergence, $[\log t]_0^1$ for $x = 0$, which diverges logarithmically, and $[t^x/x]_0^1$ for $x < 0$, which diverges like a power.
(ii) $\Gamma(x+1) = -\int_0^\infty t^x d(e^{-t}) = -[t^x e^{-t}]_0^\infty + \int_0^\infty x t^{x-1} e^{-t} dt = x\Gamma(x)$ for $x > 0$.
(iii) So $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-2) = \ldots = n!$ Since the factorial has non-negative integer arguments and Γ has positive arguments, Γ gives a continuous extension of the factorial.
(iv) $\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} dt$. Putting $t = \frac{1}{2}x^2$, $dt = xdx$, the right is

$$\int_0^\infty x^{-1}.\sqrt{2}.e^{-x^2/2}.xdx = \sqrt{2}\int_0^\infty e^{-x^2/2}dx = \frac{1}{\sqrt{2}}\int_{-\infty}^\infty e^{-x^2/2}dx.$$

So
$G^2(1/2) = \frac{1}{2}\int_{-\infty}^\infty e^{-x^2/2}dx.\int_{-\infty}^\infty e^{-y^2/2}dy$
$= \frac{1}{2}\int\int_{plane} e^{-(x^2+y^2)/2}dxdy \quad (cartesians) = \frac{1}{2}\int\int_{plane} e^{-r^2/2}.rdrd\theta \quad (polars)$
$= \frac{1}{2}\int_0^{2\pi} d\theta \int_0^\infty e^{-r^2/2}/rdr = \frac{1}{2}.2\pi.\int_0^\infty e^{-t}dt \ (t = \frac{1}{2}r^2) = \pi$.
So $\Gamma(1/2) = \sqrt{\pi}$.
Note. This is the same proof as that the standard normal probability density $\phi(x) := e^{-x^2/2}/\sqrt{2\pi}$ is a probability density, i.e. integrates to 1.

Q2 *The Gamma function (complex case).*
(i) $t^{z-1} = e^{(z-1)\log t} = e^{(x+iy-1)\log t}$, so $|t^{z-1}| = t^{x-1}$. So the argument of Q1(i) applies: the integral for $\Gamma(z)$ converges for $\Re z = x > 0$ but diverges for $\Re z = x \leq 0$.
(ii) Again, the argument in Q1(ii) applies, to give $\Gamma(1+z) = z\Gamma(z)$ for $\Re z > 0$.
(iii) For $\Re z > -1$, $\Re(1+z) > 0$, so we can use (ii) to *define* $\Gamma(z)$ as $\Gamma(1+z)/z$. The divergence (or to use the term we introduce later, the singularity) of $\Gamma(z)$ at $z = 0$ gives a singularity of $\Gamma(1+z)$ at $z = -1$, and so a singularity of $\Gamma(z) := \Gamma(z+1)/z$ at $z = -1$.
(iv) Repeating this: we can extend $\Gamma(z)$ from $\Re z > -1$ to $\Re z > -2$ by defining $\Gamma(z)$ as $\Gamma(1+z)/z$. The singularity of $\Gamma(z)$ at $z = -1$ gives a singularity of $\Gamma(z) := \Gamma(1+z)/z$ at $z = -2$. Continuing in this way (or by induction), we can extend the domain of definition of $\Gamma(z)$ to the whole of the complex plane $\mathbf{C}$, except for singularities at $z = 0, -1, -2, \ldots, -n, \ldots$.
(v) $\Gamma(-2.5) = \Gamma(-1.5)/(-2.5) = \Gamma(-0.5)/(-1.5)(-2.5)$
$= \Gamma(0.5)/(-0.5)(-1.5)(-2.5) = -8\sqrt{\pi}/3.5 = -8\sqrt{\pi}/15$.
$\Gamma(3.5) = (2.5)\Gamma(2.5) = (2.5)(1.5)\Gamma(1.5) = (2.5)(1.5)(0.5)\Gamma(0.5)$
$= 5.3\sqrt{\pi}/8 = 15\sqrt{\pi}/8$.

Q3 *The Riemann zeta function* ($\sigma = \Re s > 1$).
(i) The series $\zeta(\sigma) := \sum_1^\infty 1/n^\sigma$ converges/diverges with the integral $I(\sigma) :=$

$\int_1^\infty dx/x^\sigma$, by the Integral Test. As in Q1(i), $\int_1^x x^{-\sigma} dx$ is $(1-1/x^{\sigma-1})/(\sigma-1) \uparrow 1/(\sigma-1) < \infty$ as $x \to \infty$ for $\sigma > 1$, $\log x \uparrow \infty$ for $\sigma = 1$, $(x^{1-\sigma}-1)/(1-\sigma) \uparrow \infty$ for $\sigma < 1$. So the series converges for $\sigma > 1$ but not for $\sigma \leq 1$.
(ii) By (i), the half-plane of convergence is $\Re s > 1$. This is also the half-plane of absolute convergence, as the coefficients are all positive (all 1).

Q4 *The Riemann zeta function* ($\sigma = \Re s > 0$).
(i) As $1/n^\sigma \downarrow 0$ $(n \to \infty)$ for $\sigma > 0$, $\sum (-)^{n-1}/n^\sigma$ converges for $\sigma > 0$ by the Alternating Series Test. It cannot converge for $\sigma \leq 0$, as then the nth term does not tend to zero. So the half-plane of convergence of the Dirichlet series $\sum_1^\infty (-)^{n-1}/n^s$ is $\Re s > 0$. Its half-plane of absolute convergence is $\Re s > 1$, as we get the series for the zeta function (Q3) on taking moduli.
(ii)

$$\sum_1^\infty (-)^{n-1}/n^\sigma = \sum_{nodd} - \sum_{neven} = \sum_o - \sum_e, \quad \text{say, and} \quad \zeta(\sigma) = \sum_1^\infty 1/n^\sigma = \sum_o + \sum_e.$$

But

$$\sum_e = \sum_{m=1}^\infty 1/(2m)^\sigma = 2^{-\sigma} \sum_{m=1}^\infty 1/m^\sigma = 2^{-\sigma}\zeta(s).$$

So $\sum_o = \zeta(\sigma) - 2^{-\sigma}\zeta(\sigma) = (1-2^{-\sigma})\zeta(\sigma)$,
$\sum_1^\infty (-)^{n-1}/n^\sigma = (1-2^{-\sigma})\zeta(\sigma) - 2^{-\sigma}\zeta(\sigma) = (1-2^{1-\sigma})\zeta(\sigma)$.
(iii) So we can use (ii) to *define*

$$\zeta(s) := \frac{1}{(1-2^{1-s})}.\sum_1^\infty (-)^{n-1}/n^s,$$

whenever (a) the numerator is defined, i.e. $\Re s > 0$, and (b) the denominator is non-zero (we cannot divide by zero), i.e. $s \neq 1$.
(iv) At $s = 1$, the numerator is

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots, = \log 2,$$

given. So as $s \to 1$, by L'Hospital's Rule,

$$(s-1)\zeta(s) \sim \log 2.\frac{s-1}{(1-2^{1-s})} = \log 2.\frac{(s-1)}{(1-exp\{-(s-1)\log 2\})} \to 1.$$

Note. 1. This process of defining a function in one way, and then extending its domain of definition in some other way as above, is very general and important. We shall meet it later in the course as ANALYTIC CONTINUATION.
2. The Riemann zeta function plays an important role in Analytic Number Thoery, and in the study of the distribution of the prime numbers p.

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