M2PM3 SOLUTIONS 1. 23.1.2009

Q1. (i) If x is a terminating decimal, x is a rational of the form $n + m/10^k$.
If x is a recurring decimal, say

$$x = n.a_1 \ldots a_k b_1 \ldots b_\ell \ldots b_1 \ldots b_\ell \ldots,$$

x is $n.a_1 \ldots a_k$ (rational, above) $+y$, where writing

$$b := b_1/10 + \ldots b_\ell/10^\ell$$

(rational, above), y is a geometric series with first term $b/10^k$ and common ratio $10^{-\ell}$. So

$$y = b.10^{-k}/(1 - 10^{-\ell}),$$

rational, so x is rational.

If x is rational, $x = m/n$ say:
(a) take off its integer part – so reducing to $0 \leq m < n$,
(b) cancel m/n down to its lowest terms.
(ii) Now find the decimal expansion of m/n by the Long Division Algorithm. Let the remainders obtained by $r_1, r_2, \ldots$. The expansion *terminates* if some $r_k = 0$. It *recurs* if some remainder has *already occurred.* As there are only $n-1$ different possible non-zero remainders, the expansion must terminate (with remainder 0) or recur (with a remainder the first repeat of one of $1, 2, \ldots, n-1$) after at most $n - 1$ places.
(The examples 1/7, 2/7, 3/7, 4/7, 5/7, 6/7 show that all $n - 1$ places may be needed.)
(iii) Similarly with 10 replaced by 2,3, ...

Q2. $f(x) = \exp(-1/x^2)$.
(i)

$$f' = -\frac{2}{x^3}e^{-1/x^2}, \qquad f'' = \frac{6}{x^5}e^{-1/x^2} - \frac{2}{x^3}.\Big(-\frac{2}{x^3}\Big)e^{-1/x^2}.$$

If inductively $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$ with P_n a polynomial (of degree $3n$),

$$f^{(n+1)}(x) = -\frac{1}{x^2}P_n'(1/x)e^{-1/x^2} - \frac{2}{x^3}P_n(1/x)e^{-1/x^2} = P_{n+1}(1/x)e^{-1/x^2},$$

completing the induction with $P_{n+1}(x) = -x^2 P_n'(x) - 2x^3 P_n(x)$ (by induction, P_n has degree $3n$).
(ii) $f(0) = 0$.

$$\frac{f^{(n+1)}(x) - f^{(n+1)}(0)}{x - 0} = \frac{1}{x}P_{(n+1)}(1/x)e^{-1/x^2} \to 0 \qquad (x \to 0),$$

as $x^{-1}P_{(n+1)}(1/x) = yP_{(n+1)}(y)$ is a polynomial in $y := 1/x$ ($\to \infty$ as $x \to 0$), $e^{-1/x^2} = e^{-y^2}$, but exponential growth is faster than polynomial growth.
(iii) The Taylor series of f about 0 at x is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} 0.x^n/n! = \sum 0 = 0,$$

for all x. This $= f(x)$ *only* at $x = 0$.
Note. In Ch. II, §7, The Cauchy-Taylor Theorem, we shall see how to avoid such bad behaviour.

Q3. (θ, ϕ) represents a point uniquely, *except* at the North Pole $\theta = 0$, when any value of ϕ will do. For, near the North Pole, we can approximate spherical polars on the sphere by plane polars on the tangent plane to the sphere at the North Pole. Plane polars (r, θ) are not unique (only) at the origin: if $r = 0$, any θ will do.

Another way of putting this is that pointing South is facing towards the South Pole. This is a unique direction at any point except the poles: at the North Pole, any direction faces South.

Q4. (i) For $x = 0$, $f_n(x) = 0 \to 0$. For $x > 0$, $f_n(x) \sim 1/(nx) \to 0$.
(ii)

$$f_n'(x) = \frac{(1 + n^2x^2)n - nx.2n^2x}{(1 + n^2x^2)^2} = \frac{n(1 - n^2x^2)}{(1 + n^2x^2)^2} = 0$$

where $x = 1/n$. One can check this is a maximum. Then

$$\sup_{[0,\infty)} f_n(.) = f_n(1/n) = \frac{1}{1+1} = \frac{1}{2},$$

which does not tend to 0 as $n \to \infty$.
(iii) $f_n \to 0$ uniformly iff $\sup_{[0,\infty)} f_n \to 0$.

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