M2PM3 PROBLEMS 7. 7.3.2009

Q1. Obtain

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{1}{2}\pi$$

by real methods, as follows. Write

$$F(t) := \int_0^\infty e^{-xt} \frac{\sin x}{x} dx \qquad (t \geq 0).$$

Assuming that one can 'differentiate under the integral sign' (one can – you may assume this), obtain

$$F'(t) = -\int_0^\infty e^{-xt} \sin x dx \qquad (t > 0).$$

Integrate by parts twice to show that

$$F'(t) = -1/(1+t^2).$$

Integrate to find $F(t)$, and use $F(t) \to 0$ as $t \to \infty$ to deduce $I := F(0+) = \pi/2$.

Q2. Find

$$I := \int_0^\infty \frac{x^2}{x^4 + 5x^2 + 6} dx.$$

Q3. Show that for $p, q \geq 0$,

$$\int_{-\infty}^\infty \frac{\cos px - \cos qx}{x^2} dx = -\pi(p - q).$$

NHB