M2PM3 PROBLEMS 6. 5.3.2009

Q1 (*Poisson kernel*). Let $z = re^{i\theta}$, $w = Re^{i\phi}$, $z \neq w$. Show that
(i)

$$\Re\Big(\frac{w+z}{w-z}\Big) = \frac{|w|^2 - |z|^2}{|w-z|^2};$$

(ii) $|w-z|^2 = R^2 - 2Rr\cos(\theta - \phi) + r^2$;
(iii)

$$\Re\Big(\frac{w+z}{w-z}\Big) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\phi) + r^2}.$$

Q2 (*Poisson integral*). Let f be holomorphic on the closed disc $\bar{D}(0,R) := \{z : |z| \leq R\}$, $z = re^{i\theta} \in D(0,R) = \{z : |z| <\}$. By applying the Cauchy integral formula to fg, where $g(w) := (R^2 - r^2)/(R^2 - w\bar{z})$, or otherwise, show that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\phi) + r^2} . f(Re^{i\phi}) d\phi.$$

Q3 *Schwarz's Formula.* (i) If u is a harmonic function, and u is known on the circle $C(0,R)$, show that

$$f(z) := \frac{1}{2\pi i}\int_{C(0,R)} \Big(\frac{w+z}{w-z}\Big) u(w) \frac{dw}{w} + iC$$

(C an arbitrary real constant) is holomorphic in $|z| < R$.
(ii) Show that f has real part u, where u is given inside the disc $D(0,R)$ in terms of its values on the boundary by

$$u(z) = \frac{1}{2\pi i}\int_{C(0,R)} \Re\Big(\frac{w+z}{w-x}\Big) u(w) \frac{dw}{w}.$$

(iii) Deduce that

$$u(z) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{u(Re^{i\phi})}{R^2 - 2Rr\cos(\theta-\phi) + r^2} d\phi \qquad (z = re^{i\phi}).$$

(iv) Deduce that the values of a holomorphic function f in a disc D are specified by the values of its real part u on the boundary.
(v) Deduce that the values of f in D are also specified by the boundary values of its imaginary part v.

Q4 (*Liouville's theorem on* $\mathbf{C}^*$). We say that $f(z)$ has a property *at infinity* if $f(1/z)$ has the property at 0. Show that if f is holomorphic in the extended complex plane $\mathbf{C}^*$ (i.e., entire – holomorphic in $\mathbf{C}$ – and holomorphic at ∞), f is constant.

So a non-constant entire function has a singularity at ∞. Give some examples.

NHB