M2PM3 PROBLEMS 4. 12.2.2009

Q1. *Triangle Lemma.*

Let Δ be a triangle in $\mathbf{C}$ with perimeter of length L. Show that if z_1, z_2 are points inside or on Δ,

$$|z_1 - z_2| \leq L.$$

[This is "obvious", in that it is geometrically clear – the point is that you are asked for a proof. Reason: this is needed in the proof of Cauchy's Theorem for Triangles.]

Q2. *Holomorphy and Conjugation.*

If $f(z)$ is holomorphic, show that $f(\bar{z})$ is not holomorphic.

Q3. *Unions of Domains.*

If D_i are domains and their intersection $\bigcap_i D_i$ is non-empty, show that their union $\bigcup_i D_i$ is a domain [i.e., is connected, as it is non-empty and open].

[If D_1, D_2 are domains with empty intersection, their union $D_1 \cup D_2$ is disconnected, by definition of disconnected, so is not a domain. So the condition of non-empty intersection is essential here.]

Q4. *Connected Components.*

A connected subset of a set S in the complex plane (or any topological space) is *maximal* if it is not a proper subset of any larger connected subset. The maximal connected subsets of S are called the *(connected) components* of S. Show (by considering all connected subsets of S containing z and using Q3, or otherwise) that each $z \in S$ belongs to a unique (connected) component of S.
Note. (i) A connected set S is called *simply connected* if its complement S^c has one connected component, *doubly connected* if it has two, *n-ply connected* if it has n.
(ii) We shall see that simply connected sets really are simpler in Complex Analysis, in connection with Cauchy's Theorem.

Q5. Where are the following power series holomorphic [i.e., what are their circles of convergence]?
(i) $\sum_{n=1}^{\infty} (-)^n z^n/n$,
(ii) $\sum_{n=0}^{\infty} z^{5n}$,
(iii) $\sum_{n=0}^{\infty} z^n/n^n$?

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