M2PM3 PROBLEMS 2. 22.1.2009

Q1. If $f : X \to Y$, for $A \subset X$ the *image* $f(A)$ of A under f is $\{f(x) : x \in A\}$, and for $B \subset Y$ the *inverse image* $f^{-1}(B)$ of B under f is $\{x : f(x) \in B\}$.

In a topological space, or metric space, we call f *continuous* if inverse images of open sets (under f) are open. Show that this agrees with the ϵ, δ definition you know already in the cases ($\mathbf{R}^d$, $\mathbf{C}$) you have studied already.

Q2. Show that f is continuous iff inverse images of closed sets are closed.

Q3. Show that the continuous image of a compact set is compact: if $f : X \to Y$ is continuous, $A \subset X$ is compact, then $f(A) \subset Y$ is compact.

Q4. Deduce that if $f : [a, b] \to \mathbf{R}$ is continuous, f is bounded.

Q5. (i) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = \frac{\pi}{a} \qquad (a > 0).$$

(ii) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{ab(a + b)} \qquad (a, b > 0).$$

Hint: use partial fractions to reduce to (i) if $a \neq b$. Then use this and continuity if $a = b$. We shall return to this example in Ch. III by residue calculus.

Q6. (i) Show that if

$$F(t) := \int_0^{\infty} e^{-x} \cos xt dx,$$

$$F(t) = 1/(1 + t^2).$$

Hint: integrate by parts twice.
(ii) Deduce that

$$\int_{-\infty}^{\infty} e^{ixt} . \frac{1}{2} e^{-|x|} dx = \frac{1}{(1 + t^2)}.$$

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