Postscript to Lecture 33, 27.3.2009

$$(i) \qquad \int_0^\infty \frac{\log x}{1+x^2} dx = 0, \qquad (ii) \qquad \int_{-\infty}^\infty \frac{ue^u}{1+e^{2u}} du = 0.$$

These follow easily by Real Analysis, as below (we give essentially the same calculation twice, for completeness).

(i) Put $I := \int_0^\infty$, $I_1 := \int_0^1$, $I_2 := \int_1^\infty$. In I_1, put $y := 1/x$ (so $x = 1/y$):

$$I_1 = \int_\infty^1 \frac{\log(1/y)(-dy/y^2)}{1+1/y^2} = -\int_1^\infty \frac{\log y}{y^2+1} dy,$$

multiplying top and bottom by y^2. But this is $-I_2$ (switching from y to x as integration variable). So $I = I_1 + I_2 = -I_2 + I_2 = 0$.

(ii) Put $I := \int_{-\infty}^\infty$, $I_1 := \int_{-\infty}^0$, $I_2 := \int_0^\infty$. In I_1, put $v := -u$:

$$I_1 = \int_\infty^0 \frac{(-v)e^{-v}(-dv)}{1+e^{-2v}} = -\int_0^\infty \frac{ve^v}{e^{2v}+1} dv,$$

multiplying top and bottom by e^{2v}. But this is $-I_2$ (writing u for v), and we finish as before.

The keyhole contour. Recall its use in III.8 to evaluate $I := \int_0^\infty x^{a-1}dx/(1+x)$:

(a) We *cut* the plane, deleting the positive real axis: it is now impossible to go right round the origin, and this has the effect of making the many-valued complex power z^a (which has a singularity at the origin – a *branch-point*) single-valued.

(b) On the upper part of the cut, $z = x$, $z^{a-1} = x^{a-1}$, contributing I to the contour integral. On the lower part of the cut, $z = xe^{2\pi i}$, $z^{a-1} = x^{a-1}e^{2\pi i(a-1)}$, contributing $Ie^{2\pi i(a-1)}$, hence the answer as in lectures.

A change of variable $x \mapsto \log x$ in I (as in going from (i) to (ii) above) maps the upper edge of the cut to the real line. It maps the lower edge of the cut to the line $y = 2\pi$, since $\log(xe^{2\pi i}) = \log x + 2\pi i$. The small circle in the keyhole contour (joining up the left-hand ends of the upper and lower edges of the cut) corresponds to a vertical line joining $-R$ to $-R+2\pi i$, and similarly the large circle in the keyhole (joining the right-hand ends) corresponds to the line from R to $R+2\pi i$. So we have some choice: anything we can do with a keyhole contour (and a many-valued integrand), we can do instead with a large rectangle with vertices $\pm R$, $\pm R + 2\pi i$ (and a single-valued integrand) – recall III.3, Translation of the line of integration. It's worth remembering this link between the keyhole and this rectangle – and deciding which route

you prefer.
Indented semicircle. Recall the indented semicircle used in lectures to evaluate $\int_0^\infty (\sin x/x)dx = \pi/2$. We indent to avoid the pole at the origin.

This indented semicircle is also useful for $\int_0^\infty (\log x)^2 dx/(1+x^2)$, as in Lecture 33 (the integral of which (i) above is a simpler version). Just as with the keyhole contour, the indented semicircle corresponds under the change of variable $x \mapsto \log x$ (which gets rid of the logarithm, which is many-valued, and has a *branch-point* at the origin) to the rectangle with vertices $\pm R$, $\pm R + \pi i$. Again, it is worth remembering this link – and again, deciding which route you prefer.
Note. 1. We couldn't use the indented semicircle for $I := \int_0^\infty x^{a-1} dx/(1+x)$ because the contour would then go through the pole at $z = -1$!.
2. We couldn't use the keyhole for (i) above. For, on the lower edge of the cut, $\log(x + 2\pi i) = \log x + 2\pi i$, and the limits from $+\infty$ to 0 would mean that the I contributions we want would cancel out between the upper and lower edges of the cut. By contrast, using the indented semicircle, on the negative real axis $z = re^{\pi i} = -r$, $\log z = \log r + \pi i$, so we get (writing $\mathbf{R}_-$ for the negative real axis, traversed left to right)

$$\int_{\mathbf{R}_-} \frac{\log z}{1+z^2} dz = \int_\infty^0 \frac{\log(-r)}{1+r^2}(-dr) = \int_0^\infty \frac{\log r + \pi i}{1+r^2} dr$$

$$= \int_0^\infty \frac{\log x}{1+x^2} dx + \pi i \int_0^\infty \frac{1}{1+x^2} dx = I + \pi i(\pi/2).$$

On the positive real axis we get I. So now the two I terms we want are both present, rather than cancelling out.
Contours and choice of contour.

Recall the contours we have used: circles and ellipses; triangles, squares, rectangles; sectors; semicircles; indented versions of any of these; keyholes.

Recall also (see lecture notes at the beginning of Chapter III) that selection of a contour is not an automatic process, but is a matter of experience. This is just the same situation as when learning integration in the Sixth Form: selection of method (parts? substitution? if so, what substitution? ...) is a matter of experience, trial and error, ..., just as then.

During revision, you should familiarise yourselves with the range of examples we have covered, in lectures and on the problems/solutions. Under exam conditions, you will not be asked to go beyond things similar to examples you have met before. NHB