lecture10.tex 2.2.2009 [Department's lectures cancelled – snow]

$$\tan\ z := \sin\ z/\cos\ z.$$

Note. 1. This has singularities (infinities) at the zeros of cos z, as in the real case – that is, at points $z = (n + \frac{1}{2})\pi$, n integer. We shall see later how to classify such singularities.
2. Recall in the real case how, as we approach $\pi/2$ from below, the graph of tan goes off to $+\infty$, and then reappears from $-\infty$ as we go through $\pi/2$. This suggests that there is a sense in which $+\infty$ and $-\infty$ are "the same" (even though, taking the ordering of the real line into account, they are "as far apart as they could be"). This is true; the sense is that of Alexandrov (one-point) compactification, which we met via stereographic projection.
What is π?

We first meet π defined as the ratio of the circumference of a circle to its diameter. (This overlooks the need to prove that this ratio is the same for all circles, but let that pass.) We then meet π in elementary trigonometry, in connection with the functions sin, cos and tan. Now that we are defining sin and cos by their power series (and tan by their ratio), we need a new definition of π. We define $\pi/2$ to be the smallest positive root of the real function cos x. It can be shown that π (and sin, cos, tan) thus defined is consistent with what we know already. This should be in all the books, but isn't. One source is Appendix 4, p.584–7, of
E. T. Whittaker and G. N. Watson, *Modern Analysis*, 4th ed., CUP, 1946.
4. Hyperbolic functions. As before,

$$ch\ z := \frac{1}{2}(e^z + e^{-z}), \qquad sh\ z := \frac{1}{2}(e^z - e^{-z}), \qquad th\ z := sh\ z/ch\ z.$$

Then $ch\ z = \cos\ iz$, $i\ sh\ z = \sin\ iz$.
5. Logarithms.

In the real case, the logarithm is the inverse function of the exponential function: for x real, $\log x = y$ means $e^y = x$. This extends to the complex case, with one complication. For complex z, w, $\log z = w$ means $e^w = z$. But since $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + i.0 = 1$, $e^{2k\pi i} = 1^k = 1$ for any integer k. So if $e^w = z$, then also $e^{w+2\pi ki} = z$. So if $\log z = w$, then also $\log z = w + 2\pi ki$: the complex logarithm is (infinitely) *many-valued.* It is thus NOT a function, which must be *single-valued.*

The logarithm changes its value when z winds round the origin (completes a rotation around 0). One way to obtain single-valuedness is to prevent this, by introducing a *cut*. For instance, if we remove the negative real axis $(-\infty, 0)$ from the complex plane, one can define a single-valued logarithm on the resulting cut plane.

Another way to make log single-valued is to make the argument *arg* single-valued, by restricting θ in $z = |z|e^{i\theta}$ ($\theta = arg\ z$) to, e.g., $\theta \in (-\pi, \pi]$. This gives the it principal value of *arg*, or log. But (as with the argument, in Ch. I) this procedure is both arbitrary and discontinuous.

Complex nth roots of unity. For integer k, $1 = e^{2\pi ik}$, so

$$(e^{2\pi ik/n})^n = e^{2\pi ik} = 1.$$

One can reduce to $k \in \{0, 1, 2, \ldots, n-1\}$ without loss. This gives the n *complex nth roots of unity.* In the Argand diagram, they correspond to the vertices of a regular n-gon (n-sided polygon) with vertices equally spaced out on the unit circle and one vertex at $z = 1$. As

$$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \ldots + z + 1),$$

$z = 1$ is one root (real), and the other $n - 1$ roots ω satisfy

$$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0.$$

The nth roots of unity form an abelian group under multiplication, (isomorphic to) the *cyclic group of order* n. [See Exam 2008 for a question on them.]

6. Complex powers. For $a > 0$, general real powers are defined by

$$a^x := e^{x \log a}.$$

This extends to complex powers: $z^w := \exp(w \log z)$, or $e^{w \log z}$. This is *many-valued*, as log is. Similarly, $(z - z_0)^w = e^{w \log(z - z_0)}$. Here z_0 is a singularity ("point of bad behaviour"), called a *branch-point.*

Because the ambiguity of value is of the simple type "$+2\pi ki$", one can avoid many- valuedness of such non-functions f by regarding them as single-valued functions, taking values not in the complex plane, but in a space R, visualised as an infinite stack of complex planes (the *sheets*), appropriately connected or spliced together so that as we increase k by going round the origin, we rise up to the next sheet. Such an R is a *Riemann surface* (G. F. B. Riemann (1826-66) in 1851; Felix Klein (1849-1925) in 1882).