M2PM3 COMPLEX ANALYSIS: SOLUTIONS TO EXAMINATION, 2008

Q1. (i) $(1+2i)^2 = 1+4i-4 = -3+4i$.
The quadratic $z^2+2iz+2-4i$ has roots

$$z = \frac{-2i \pm \sqrt{-4-4(2-4i)}}{2},$$

i.e. $-i \pm \frac{1}{2}\sqrt{-4-8+16i} = -i \pm \frac{1}{2}\sqrt{-12+16i} = -i \pm \sqrt{-3+4i} = -i \pm (1+2i)$, giving roots

$$1+i \quad \text{or} \quad -1-3i.$$

(ii) The roots are where $z^4 = -1 = e^{i\pi} = e^{(2n+1)i\pi}$. These are $z = e^{i\pi/4}$ $(n=0)$, $e^{3i\pi/4}$ $(n=1)$, $e^{5i\pi/4} = e^{-3i\pi/4}$ $(n=2)$, $e^{7i\pi/4} = e^{-i\pi/4}$ $(n=3)$.

In the Argand diagram, these are the points on the unit circle with arguments $\pm\pi/4$, $\pm 3\pi/4$ (the vertices of a square with diagonals $y = \pm x$).
(a) The complex factorization into four linear factors is

$$z^4+1 = (z-e^{i\pi/4})(z-e^{-i\pi/4})(z-e^{3i\pi/4})(z-e^{-3i\pi/4}).$$

(b) The real factorization into two quadratics is

$$z^4+1 = (z^2-2z\cos\pi/4+1)(z^2-2z\cos 3\pi/4+1) = (z^2-\sqrt{2}z+1)(z^2+\sqrt{2}z+1).$$

(iii) With γ the ellipse $x^2/a^2+y^2/b^2=1$ parametrized by $x = a\cos\theta$, $y = b\sin\theta$, $\int_\gamma dz/z = 2\pi i$ by Cauchy's Residue Theorem, since $1/z$ has residue 1 at 0.
So as $z = a\cos\theta + ib\sin\theta$ gives $dz = (-a\sin\theta + ib\cos\theta)d\theta$,

$$2\pi i = \int_0^{2\pi} \frac{-a\sin\theta + ib\cos\theta}{a\cos\theta + ib\sin\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(-a\sin\theta + ib\cos\theta)(a\cos\theta - ib\sin\theta)}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(b^2-a^2)\sin\theta\cos\theta + iab(\cos^2\theta + \sin^2\theta)}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta.$$

Equating imaginary parts,

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta = 2\pi,$$

whence the result on dividing by ab.
[All unseen]
Note: $z = x+iy = a\cos\theta + ib\sin\theta$ from the hint; z (at least) is needed in the denominator, to get a and b there; $dz = (-a\sin\theta + ib\cos\theta)d\theta$. So we need (at least) $dz/z = [(-a\sin\theta + ib\cos\theta)/(a\cos\theta + ib\sin\theta)]d\theta$. On multiplying top and bottom by $(a\cos\theta - ib\sin\theta)$ to make the denominator real, it turns out that this is all we need (see above). A direct attack using $z = e^{i\theta}$ is possible, but this is harder – it is geared to the unit circle, not the ellipse as here. [Note that the result is immediate when $a = b$ and the ellipse is a circle.]

Q2. (i) The Cauchy-Riemann equations (C-R) are $u_x = v_y$, $u_y = -v_x$.
(ii) A *harmonic* function is one for which the Laplacian $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$ vanishes.
Differentiating partially wrt x:

$$\begin{aligned} u_{xx} &= v_{yx} && \text{(C-R)} \\ &= v_{xy} && \text{(by continuity of partials: } f \text{ is holomorphic)} \\ &= -u_{yy} && \text{(C-R).} \end{aligned}$$

So $\Delta u = u_{xx} + u_{yy} = 0$, and u is harmonic. Similarly, v is harmonic.
(iii) Given u harmonic, integrate $v_y = u_x$ wrt y:

$$v = \int u_x dy + g(x)$$

(the additive constant of integration may involve x). Differentiate wrt x:

$$v_x = (\partial/\partial x)(\int u_x dy) + g'(x).$$

As $v_x = -u_y$, this gives

$$g'(x) = -u_y - (\partial/\partial x)(\int u_x dy).$$

Integrate to find g, hence v, hence $f = u + iv$.
(iv) $u = x^3 - 3xy^2$: $v_y = u_x = 3x^2 - 3y^2$. Integrate wrt y: $v = 3x^2y - y^3 + g(x)$. Differentiate wrt x: $v_x = 6xy + g'(x) = -u_y = 6xy$. So $g' = 0$, $g =$ constant, c (real), w.l.o.g. 0, $v = 3x^2y - y^3$. $f = u + iv = x^3 + 3ix^2y - 3xy^2 - iy^3 = (x + iy)^3$: $f(z) = z^3$.
(v) $u = x/(x^2 + y^2)$: $u_y = -2xy/(x^2 + y^2)^2 = -v_x$ (simpler than u_x), $v_x = 2xy/(x^2 + y^2)^2$. So

$$v = y \int \frac{2x}{(x^2+y^2)^2} dx + g(y) = y \int \frac{d(x^2)}{(x^2+y^2)^2} + g(y) = -\frac{y}{(x^2+y^2)} + g(y),$$

$$v_y = -\frac{1}{(x^2+y^2)} + \frac{2y^2}{(x^2+y^2)^2} + g'(y) = \frac{y^2 - x^2}{(x^2+y^2)^2} + g'(y).$$

By C-R,

$$v_y = u_x = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}.$$

Comparing, $g' = 0$, $g = c$ constant, w.l.o.g. 0. So

$$v = -\frac{y}{x^2+y^2}, \qquad f = u + iv = \frac{x - iy}{x^2+y^2} = \frac{1}{x - iy} : \quad f(z) = 1/z.$$

(vi) As $f(z) = z^3$ is holomorphic everywhere (entire), its real part u in (iv) is harmonic, being the real part of a holomorphic function. As $f(z) = 1/z$ is holomorphic except at 0, its real part u in (v) is likewise harmonic except at the origin. [(i) – (iii): Lectures; (iv) – (vi): Unseen]

Q3. (i) Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)} dz.$$

(ii) For $r > 0$ so small that the circle $\gamma(a, 2r)$ with centre a and radius $2r$ is contained in the interior $I(\gamma)$ of γ,

$$\int_\gamma \frac{f(z)}{(z-a)} dz = \int_{\gamma(a,2r)} \frac{f(z)}{(z-a)} dz,$$

by the Deformation Lemma. Similarly, for $|h| < r$, with $a+h$ in place of a. Combining,

$$f(a+h) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a-h)} dz, \qquad f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)} dz,$$

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{1}{2\pi i h} \int_{\gamma(a,2r)} f(z) \Big(\frac{1}{z-a-h} - \frac{1}{z-a} \Big) dz \\ &= \frac{1}{2\pi i} \int_{\gamma(a,2r)} \frac{f(z)}{(z-a-h)(z-a)} dz \\ &\to \frac{1}{2\pi i} \int_{\gamma(a,2r)} \frac{f(z)}{(z-a)^2} dz \qquad (h \to 0), \end{aligned}$$

estimating the difference between the two integrands on the right using

$$\frac{1}{(z-a-h)(z-a)} - \frac{1}{(z-a)^2} = \frac{h}{(z-a)^2(z-a-h)} \sim \frac{h}{(z-a)^2} \qquad (h \to 0)$$

and the ML Inequality.
So $f'(a)$ exists, $=$ RHS.
We may replace $\gamma(a, 2r)$ by γ in RHS, by the Deformation Lemma, giving

$$f'(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^2} dz.$$

(iii) Similarly,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{(n+1)}} dz.$$

(iv) With $\gamma = \gamma(a, R)$, $|f^{(n)}(a)| \le n! M/R^n$ follows by the ML Inequality.
(v) $f(z)$ is *entire* if it is holomorphic throughout $\mathbf{C}$ (i.e., no singularities in the complex plane).
(vi) If f is entire and $|f(z)| \le c|z|^k$ for some c and $|z|$ large, by (iv) with M replaced by cR^k, $|f^{(n)}(a)| \le n! c R^k / R^n$. Let $R \to \infty$: for $n > k$, $f^{(n)}(a) = 0$ for each a, so $f^{(n)} \equiv 0$, so f is a polynomial of degree at most k.
[Seen: (i) – (v) in Lectures, (vi) in Problems 6]

Q4. (i) Use as contour the unit circle γ, parametrized by $z = e^{i\theta}$. So $d\theta = dz/iz$, $\cos\theta = (z + z^{-1})/2$.
To show:

$$I := \int_0^{2\pi} \frac{e^{3i\theta}}{5 - 4\cos\theta} d\theta = \frac{\pi}{12}. \qquad (*)$$

$$I = \int_\gamma \frac{z^3}{iz[5 - 2(z + 1/z)]} dz.$$

The denominator in the integrand is

$$iz[5-2(z+1/z)] = i[-2+5z-2z^2] = -i[2z^2-5z+2] = -i(2z-1)(z-2) = -2i(z-\frac{1}{2})(z-2).$$

So

$$I = \frac{i}{2} \int_\gamma \frac{z^3}{(z - \frac{1}{2})(z - 2)} dz = \frac{i}{2} \int_\gamma f(z) dz,$$

say. Now $f(z) = z^3/[(z - 1/2)(z - 2)]$ has simple poles at $z = 1/2$, $z = 2$, only the first of which is inside γ.
By the Cover-Up Rule,

$$Res_{1/2} f = \frac{(1/2)^3}{[(1/2) - 2]} = 1/[8(-3/2)] = -1/12.$$

So by Cauchy's Residue Theorem, $I = (1/2).2\pi i.(-1/12) = \pi/12$,
proving $(*)$. Now take real and imaginary parts.
Note. The cosine integral can be evaluate separately, using $\cos 3\theta = (1/2)(z^3 + z^{-3})$, but this is harder as it introduces a triple pole at $z = 0$. The sine integral is clearly 0 (use limits $-\pi$ and π: odd integrand, symmetrical limits).
(ii) Use as contour γ the interval $\gamma_1 := [-R, R]$ closed by the semi-circle γ_2 on this base in the upper half-plane, and as function $f(z) := 1/(z^2 + 1)^2$.
This has double poles at $z = \pm i$, only $z = i$ being inside γ.

$$\int_{\gamma_1} f = \int_{-R}^{R} \to I : \int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2} \qquad (R \to \infty),$$

the required integral. As $|f| = O(1/R^4)$ on γ_2, $\int_{\gamma_2} f = O(1/R^3) \to 0$ as $R \to \infty$, by the ML Inequality.
For z near i, write $z = i + \zeta$, ζ small. Then $z^2 + 1 = (i + \zeta)^2 + 1 = -1 + 2i\zeta + \zeta^2 + 1 = 2i\zeta + \zeta^2$, so

$$f(z) = \frac{1}{(1 + z^2)^2} = \frac{1}{(2i\zeta + \zeta^2)^2} = \frac{1}{\zeta^2}.\frac{1}{(-4)}.\Big(1+\frac{\zeta}{2i}\Big)^{-2} = -\frac{1}{4\zeta^2}.\Big(1-\frac{\zeta}{i}+O(\zeta^2)\Big).$$

$$Res_i f = \text{coeff. } 1/\zeta = -\frac{1}{4}.\frac{(-1)}{i} = \frac{1}{4i} = -i/4 : \quad I = 2\pi i Res_i f = 2\pi i.(-i/4) = \pi/2.$$

Combining: by Cauchy's Residue Theorem, $I = \pi/2$.
(The residue at the double pole may also be evaluated by differentiation.)
Note: The integral can be also be evaluated by real analysis: use $x = \tan\theta$.
[All unseen – similar seen]