M2PM3 COURSEWORK 2 SOLUTIONS, 5.3.2009

Q1. $\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1}e^{-t}dt.\int_0^\infty u^{y-1}e^{-u}du$. Putting $u = tv$, this gives

$$\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1}e^{-t}dt.\int_0^\infty t^{y-1}e^{-tv}v^{y-1}.tdv,$$

or changing the order of integration and writing $w := t(1+v)$,

$$\int_0^\infty v^{y-1}dv\int_0^\infty t^{x+y-1}e^{-t(1+v)}dv = \int_0^\infty w^{x+y-1}e^{-w}dw.\int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}dv.$$

As the first integral on RHS is $\Gamma(x+y)$, this gives

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}dv,$$

giving the first part. For the second part, make the change of variable $u := 1/(1+v)$; then $1-u = v/(1+v)$, $du = -dv/(1+v)^2$, and $v = 0, \infty$ correspond to $u = 1, 0$. So

$$\int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}dv = \int_0^1 (1-u)^{x-1}u^{y-1}du,$$

which gives the result as LHS, and so RHS, is symmetrical between x and y.

Q2. If X, Y have densities f, g, $X+Y$ has density h, where

$$h(x) = \int_0^x f(y)g(x-y)dy \qquad (x > 0).$$

Here $f(x) = x^{\lambda-1}e^{-x}/\Gamma(\lambda)$, $g(x) = x^{\mu-1}e^{-x}/\Gamma(\mu)$, so

$$h(x) = \int_0^\infty f(x-y)g(y)dy = \int_0^\infty \frac{(x-y)^{\lambda-1}e^{-(x-y)}}{\Gamma(\lambda)}.\frac{y^{\mu-1}e^{-y}}{\Gamma(\mu)}dy$$

$$= \frac{e^{-x}}{\Gamma(\lambda)\Gamma(\mu)}.\int_0^x (x-y)^{\lambda-1}y^{\mu-1}dy = \frac{x^{\lambda+\mu-1}e^{-x}}{\Gamma(\lambda)\Gamma(\mu)}.\int_0^1 (1-u)^{\lambda-1}u^{\mu-1}du = \frac{x^{\lambda+\mu-1}e^{-x}}{\Gamma(\lambda)\Gamma(\mu)}.B(x,y),$$

putting $y = xu$ in the integral. This is $c.x^{\lambda+\mu-1}e^{-x}$ for some constant c. So:
(i) h is a Gamma density, $\Gamma(\lambda+\mu)$, from its functional form,
(ii) $c = 1/\Gamma(\lambda+\mu)$ (this is the constant required to make the density integrate to 1, as it must).

The result follows on equating the two expressions for the constant c.

Q3 [**4**]. (i) [**1**] For f holomorphic, $f = u + iv$, u and v are differentiable w.r.t. x and y (as in lectures: for $\partial/\partial x$, take the difference $z - z_0$ real; for $\partial/\partial y$, take it imaginary).
(ii) [**1**] $f_x = u_x + iv_x$, $f_y = u_y + iv_y$, so

$$\partial f/\partial z := \frac{1}{2}(f_x - if_y) = \frac{1}{2}[(u_x + iv_x) - i(u_y + iv_y)] = \frac{1}{2}(u_x + v_y) + \frac{1}{2}i(v_x - u_y).$$

By the Cauchy-Riemann equations, this is $u_x + iv_x$, $f'(z)$ (lectures).
(iii) [**1**]

$$\partial f/\partial \bar{z} := \frac{1}{2}(f_x + if_y) = \frac{1}{2}[(u_x + iv_x) + i(u_y + iv_y)] = \frac{1}{2}(u_x - v_y) + \frac{1}{2}i(v_x + u_y).$$

By the Cauchy-Riemann equations, this is 0.
(iv) [**1**] As above in (iii), $\partial f/\partial \bar{z} = 0$ is equivalent to the Cauchy-Riemann equations. This and continuity of partials gives differentiability, i.e. holomorphy, as in lectures.

Q4 [**4**]. $d(\cot z)/dz = cosec^2 z$ [**1**]. So as the unit circle is closed,

$$\int_{C(0,1)} cosec^2 z dz = \int_{C(0,1)} \frac{d}{dz} \cot z dz = \int_{C(0,1)} d \cot z = [\cot z]_{C(0,1)} = 0,$$

by the Fundamental Theorem of Calculus [**2**].

Cauchy's Theorem does *not* apply, as $cosec^2 z$ has a singularity at 0 (a double pole) [**1**]. [Cauchy's Residue Theorem does apply (the residue is 0 as the pole is double rather than single) – but the lecture for this is after the deadline!]

Q5 [**4**]. Parametrize $C(0,1)$ by $e^{i\theta}$, $0 \le \theta \le 2\pi$. For $f(z) = (Im\ z)^2$, $z = e^{i\theta}$, $f(z) = \sin^2\theta$ [**1**], so the integral is

$$I = \int_0^{2\pi} \sin^2\theta . ie^{i\theta} d\theta = -\int_0^{2\pi} \sin^3\theta d\theta + i\int_0^{2\pi} \cos\theta \sin^2\theta dt = I_1 + iI_2, \text{say } [\mathbf{1}].$$

$I_1 = \int_0^{2\pi}(1 - \cos^2\theta) d\cos\theta = [\cos\theta - \frac{1}{3}\cos^3\theta]_0^{2\pi} = 0$, by periodicity of cos. Similarly, $I_2 = \int_0^{2\pi} \sin^2\theta d \sin\theta = \frac{1}{3}[\sin^2\theta]_0^{2\pi} = 0$ [**1**].
Cauchy's Theorem does not apply since $Im\ z$ is not holomorphic [**1**].

NHB