M2PM3 ASSESSED COURSEWORK 2, 2009

Set Th 26 February 2009; deadline NOON, Wed 4 March 2009

Q1 [**4**]. *Euler's Beta integral for the Gamma function: Analysis.*
Recall that $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$ $(x > 0)$. Show that for $x, y > 0$,

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x,y) := \int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}dv = \int_0^1 u^{x-1}(1-u)^{y-1}du$$

$(B(x,y)$ is called the *Beta function*). In particular, as $\Gamma(1) = 0! = 1$ and the LHS is symmetrical in x and y:

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}}{(1+v)}dv \qquad (0 < x < 1).$$

[We shall use this in Ch. III to prove $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$.]
Suggested method. Write $\Gamma(x)\Gamma(y)$ as a product of integrals over $(0,\infty)$, in t and u say. Change integration variables to v and w, where $u = tv$, $t(1+v) = w$, and interchange the order of integration.

Q2 [**4**]. *Euler's Beta integral for the Gamma function: Probability.*
We quote that
(i) A non-negative function $f(x)$ on the line that integrates to 1 is called a *probability density function* (or *density* for short). Some (not all) random variables have density functions.
(ii) If X, Y are independent random variables with densities f, g, then $X+Y$ has a density h, given by the *convolution formula*

$$h(x) = \int_{-\infty}^\infty f(x-y)g(y)dy.$$

The Gamma density in Probability Theory with parameter $\lambda > 0$ is defined by

$$f(x) := x^{\lambda-1}e^{-x}/\Gamma(\lambda) \qquad (x > 0), \qquad 0 \qquad (x \le 0).$$

Let X, Y be independent random variables, Gamma distributed with parameters λ, μ.
(i) [**2**]. Show that $X+Y$ is also Gamma distributed, with parameter $\lambda+\mu$.

(ii) [**2**]. Deduce that the Beta integral follows (identify the constant – a density must integrate to 1).

Q3 [**4**]. For $f(z)$ ($z = x + iy$) regarded as a function of x and y, write

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Show that
(i) [**1**] for f holomorphic in a domain D, these partial derivatives exist (so the above are well-defined);
(ii) [**1**] $\partial f/\partial z = f'$;
(iii) [**1**] $\partial f/\partial \bar{z} = 0$.
(iv) [**1**] If f has continuous partials and $\partial f/\partial \bar{z} = 0$, show that f is holomorphic.

Q4 [**4**]. For $C(0,1)$ the unit circle, show that

$$\int_{C(0,1)} cosec^2 z dz = 0.$$

Q5 [**4**]. Show that

$$\int_{C(0,1)} (Im\ z)^2 dz = 0.$$

Note. Cauchy's Theorem does not apply in either of Questions 4 or 5 – and you should say why not.

NHB