M2PM3 SOLUTIONS TO ASSESSED COURSEWORK 1, 2009

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Q1. (i)

$$\begin{aligned} a_1v_1 + \ldots a_nv_n &= s_1v_1 + (s_2 - s_1)v_1 + \ldots + (s_n - s_{n-1})v_n \\ &= s_1(v_1 - v_2) + s_2(v_2 - v_3) + \ldots + s_{n-1}(v_{n-1} - v_n) + s_nv_n. \end{aligned}$$

(ii) As $v_n \downarrow$, $v_k - v_{k+1} \geq 0$. This and $m \leq s_k \leq M$ give

$$m(v_k - v_{k+1}) \leq s_k(v_k - v_{k+1}) \leq M(v_k - v_{k+1}) \quad (k = 1, \ldots, n-1), \quad mv_n \leq s_nv_n \leq Mv_n.$$

Sum over $k = 1$ to $n - 1$: the left and right telescope. Using (i) for the middle gives

$$mv_1 \leq a_1v_1 + \ldots + a_nv_n \leq Mv_1.$$

(iii) If $|s_n| \leq M$ for all n, taking $m = -M$ in (ii) gives

$$!a_1v_1 + \ldots + a_nv_n| \leq Mv_1.$$

Q2. As $v_n \downarrow 0$: $\forall \epsilon > 0 \ \exists N$ such that for $n \geq N$, $0 \leq v_n < \epsilon$. As $|\sum_1^n a_k| \leq M$ for all n (for some M) – given –

$$|\sum_m^n a_k| = |\sum_1^n a_k - \sum_1^{m-1} a_k| \leq 2M \qquad \forall m, n \quad (m \leq n).$$

So by Q1(iii), $|\sum_m^n a_kv_k| \leq 2M\epsilon$ for all $m, n \geq N$. By Cauchy's General Principle, $\sum a_nv_n$ converges (as it is Cauchy).

Q3. As the series $\sum a_n$, its sequence $s_n := \sum_1^n a_k$ of partial sums converges. So (s_n) is bounded. As $v_n \downarrow \ell$, $w_n := v_n - \ell \downarrow 0$. So by Dirichlet's Test, $\sum a_nw_n$ converges, to c say:

$$a_1w_1 + \ldots a_nw_n \to c \qquad (n \to \infty).$$

That is

$$a_1v_1 + \ldots a_nv_n - \ell(a_1 + \ldots + a_n) \to c \qquad (n \to \infty).$$

But $a_1 + \ldots a_n \to b := \sum_1^\infty a_k$. So

$$a_1v_1 + \ldots a_nv_n \to c + \ell.b \qquad (n \to \infty),$$

i.e. $\sum a_nv_n$ converges.

Q4. (i) $|n^s| = |e^{s \log n}| = |e^{(\sigma + i\tau)\log n}| = e^{\sigma \log n} = n^\sigma$.
(ii) Absolute convergence of $\sum a_n/n^s$ depends only on $|a_n|$ and $|n^s| = n^\sigma$, so depends on s only through σ. If $\sigma_2 \leq \sigma_1$, then $|a_n/n^{s_2}| = |a_n|/n^{\sigma_2} \leq |a_n|/n^{\sigma_1} =$

$|a_n/n^{s_1}|$. So by the Comparison Test, absolute convergence for s_1 implies absolute convergence for s_2.
(iii) So any point of A^c is to the *left* of any point of A: if it were to the *right*, this would contradict (ii).
(iv) So A^c, A partition the real line, with A^c lying to the *left* of A. The two sets have as common boundary a point σ_a, the sup of A^c and the inf of A. By definition of A, A^c, the series is absolutely convergent in $Res > \sigma_a$ and not (so conditionally convergent or divergent) in $Res < \sigma_a$.
Note. 1. We make no statement about what happens for $Res = \sigma_a$ (as we shall not need this case).
2. The above construction of σ_a is called a *Dedekind cut* (or *Dedekind section*) – as in Dedekind's construction of the real line $\mathbf{R}$.

Q5 (i). $|n^s - (n+1)^{-s}| = |e^{-s\log n} - e^{-s\log(n+1)}| = |\int_{\log n}^{\log(n+1)} se^{-us}du| \leq |s|\int_{\log n}^{\log(n+1)} e^{-\sigma u}du = (|s|/\sigma)(n^{-\sigma} - (n+1)^{-\sigma})$.
(ii) By replacing a_n/n^{s_1} by a_n, we can w.l.o.g. take $s_1 = 0$. As $\sum a_n$ converges, by Cauchy's General Principle of Convergence, for all $\epsilon >$ there exists N such that for all $m, n \geq N$, $|a_m + \ldots + a_n| < \epsilon$. By partial summation, writing s_n for $\sum_1^n a_k$ as before,

$$\sum_m^n a_k/k^s = \sum_m^{n-1}(s_k - s_m)(k^{-s} - (k+1)^{-s}) + (s_n - s_m)n^{-s}.$$

So

$$|LHS| \leq \sum_m^{n-1} |s_k - s_m|.|k^{-s} - (k+1)^{-s}| + |s_n - s_m|.|n^{-s}|$$

$$< \epsilon.\frac{|s|}{\sigma}\sum_m^{n-1}(1/k^\sigma - 1/(k+1)^\sigma) + \epsilon.1/n^\sigma$$

$$= \epsilon.\frac{|s|}{\sigma}.(1/m^\sigma - 1/n^\sigma) < \frac{\epsilon|s|}{\sigma m^\sigma},$$

as the sum telescopes. So

$$|LHS| \to 0 \qquad (m \to \infty)$$

if $\sigma > 0$. This proves (ii) (recall we reduced first to $s_1 = 0$, $\sigma_1 = 0$).
(iii) This now follows as in Q4.
Note. 1. Absolute convergence implies convergence, but not conversely in general. The two half-planes may differ.
2. They do differ in the case of $\sum_{n=1}^\infty (-)^{n-1}/n^s$, which has $\sigma_a = 1$ and $\sigma_c = 0$. This series is important in connection with the *Riemann zeta function* $\sum_{n=1}^\infty 1/n^s$ of Analytic Number Theory; see Problems 3.

NHB