M2PM3 ASSESSED COURSEWORK 1, 2009

Set Th 28 Jan 2009. Deadline noon Wed 4 Feb 2009. 20 marks.

Q1. *Abel's Lemma, or Partial Summation.*

Write $s_n := a_1 + \ldots + a_n$. Show that
(i) $a_1 v_1 + \ldots + a_n v_n = s_1(v_1 - v_2) + \ldots + s_{n-1}(v_{n-1} - v_n) + s_n v_n$.
(ii) If $m \le a_1 + \ldots + a_n \le M$ for all n, and v_n is positive and decreasing, then $m v_1 \le a_1 v_1 + \ldots + a_n v_n \le M$.
(iii) If $|s_n| \le M$ for all n, then $|a_1 v_1 + \ldots a_n v_n| \le M v_1$ for all n.

Q2. *Dirichlet's Text for Convergence.*

If (a_n) has bounded partial sums $s_n = \sum_1^n a_k$ and $v_n \downarrow 0$, then $\sum a_n v_n$ is convergent.

Q3. *Abel's Test for Convergence.*

If $\sum a_n$ converges and $v_n \downarrow \ell$ for some ℓ, then $\sum a_n v_n$ converges.

Q4. *Dirichlet series: Half-plane of absolute convergence.*

For $s \in \mathbf{C}$, write $s = \sigma + i\tau$.
(i) Show that $|n^s| = n^\sigma$.

A series of the form $\sum_{n=1}^\infty a_n/n^s$ (the a_n can be complex) is called a *Dirichlet series.*
(ii) Show that absolute convergence of the series at s depends only on $\sigma = Res$. Show also that if the series is absolutely convergent for $s_1 = \sigma_1 + i\tau_1$, it is absolutely convergent for any $s_2 = \sigma_2 + i\tau_2$ with $\sigma_2 \ge \sigma_1$.
(iii) Let A (for 'absolute convergence') be the set of real σ for which $\sum_1^\infty a_n/n^s$ converges absolutely for s with $Res = \sigma$, $A^c := \mathbf{R} \setminus A$ be its complement. Show that any point of A^c is to the *left* of any point of A.
(iv) Deduce that if $\sigma_a := \sup\{\sigma : \sigma \in A^c\}$, then also $\sigma_a = \inf\{\sigma : \sigma \in A\}$, that $\sum a_n/n^s$ is absolutely convergent for $\sigma \in A$, and not absolutely convergent for $\sigma \in A^c$. ($\{s : Res > \sigma_a\}$ is called the *half-plane of absolute convergence.*)

Q5. *Dirichelt series: Half-plance of convergence.*
(i) Show that $|n^{-s} - (n+1)^{-s}| \le (|s|/\sigma)(n^{-\sigma} - (n+1)^{-\sigma})$ $(\sigma > 0)$.
(Hint: consider $\int_{\log n}^{\log(n+1)} se^{-us} du$.)
(ii) Show that if $\sum a_n/n^s$ converges for $s_1 = \sigma_1 + i\tau$, then it converges for all $s = \sigma + i\tau$ with $\sigma > \sigma_1$. (Hint: Reduce to $\sigma_1 = 0$ by replacing a_n by a_n/n^{σ_1}, and then use partial summation.
(iii) Deduce (as in Q4) that there is a half-plane $\sigma > \sigma_c$ of convergence, with divergence in $\sigma < \sigma_c$. ($\{s : Res > \sigma_c\}$ is called the *half-plane of convergence.*)

NHB