M2PM3 HANDOUT: VARIOUS FORMS OF CAUCHY'S THEOREM

In all versions, f is holomorphic in a domain D containing a contour γ.

Cauchy's Theorem for Triangles. If γ is a triangle and D contains (γ and) its interior, then $\int_\gamma f = 0$.

Proof. Repeated bisection.
Cauchy's Theorem for Rectangles. If γ is a rectangle and D contains (γ and) its interior, then $\int_\gamma f = 0$.

Proof. Bisect to reduce to triangles.

Similarly, Cauchy's Theorem extends to *polygons*, which can be triangulated.

The Theorem of the Primitive, and Cauchy's Theorem, can be proved for *star domains*, and in particular *convex* domains – such as discs – by considering $F(z) := \int_{[z_0,z]} f$ with z_0 a star centre for D.

Other versions (not examinable):

1. *Via Green's Theorem.*

We quote the two-dimensional form of *Green's Theorem* (George GREEN (1798-1841), *Essay*, 1828).

THEOREM. For D a domain, γ its boundary, P, Q, $\partial P/\partial y$, $\partial Q/\partial x$ continuous in D,

$$\int_\gamma P dx + Q dy = \int\int_D (\partial Q/\partial x - \partial Q/\partial y) dx dy.$$

Taking $f = u + iv$, $dz = dx + idy$, and writing $I(\gamma)$ for the interior of γ,

$$\int_\gamma f = \int_\gamma (u + iv)(dx + idy) = \int_\gamma (udx - vdy) + i \int_\gamma (udy + vdx),$$

which by Green's Theorem is

$$\int\int_{I(\gamma)} (-v_x - u_y) dx dy + i \int\int_{I(\gamma)} (u_x - v_y) dx dy,$$

which is 0 by the Cauchy-Riemann equations.

This was Cauchy's proof (1820s). Unfortunately it needs the assumption that f' is continuous.

That this assumption is superfluous was shown by E. J.-B. GOURSAT (1858-1936) in 1884. He used the *method of repeated bisection* (Goursat's Lemma), as we did in proving Cauchy's Theorem for Triangles in lectures.

2. *Via the Jordan Curve Theorem.*

This states that a simple closed curve γ in $\mathbf{C}$ (simple: non-self-intersecting) divides the plane into two connected domains, one bounded (called the *inside* of γ, $I(\gamma)$), and one unbounded (called the *outside* of γ, $O(\gamma)$). This was stated by Camille JORDAN (1838-1922) in 1866, but only proved in 1905 by Oswald VEBLEN (1880-1960). The result is topological, and the subject of Topology only emerged in the early 1900s. A complete proof of the Jordan Curve Theorem is difficult. But if we quote it, our proof of Cauchy's Theorem for Triangles extends to general contours γ such that γ and its interior $I(\gamma)$ are in D.

The reason that triangles, star domains etc. are easier here is that one can handle their interiors $I(\gamma)$ from scratch, without the Jordan Curve Theorem.

3. *Via Homotopy.*

If γ is a closed curve lying in D, γ is *homotopic to a point* in D if γ can be continuously deformed (shrunk) to a point without leaving D. For such γ, $\int_\gamma f = 0$. The homotopy condition prevents γ from winding round any "holes" in D where f fails to be holomorphic. For example, D may be an annulus (doubly connected, one hole). Then γ is prevented from winding round the hole.

3a. *Cauchy's Theorem for Simply Connected Domains.*

If D is *simply connected*, there are no holes to avoid winding round, all contours in D are homotopic to a point, and if γ is in D, so is $I(\gamma)$, and $\int_\gamma f = 0$.

4. *Via Homology.*

Drop the condition that γ be simple, and allow self-intersections. Suppose there are a number of "holes" in D, and/or a number of "singularities" (points of bad behaviour of f). Homology gives a sense in which f is essentially (i.e., topologically) "the same" as loops round these holes or singularities, described the appropriate number of times and in the appropriate sense (these are determined by the relevant winding numbers). Note that these loops are separated from each other, so do not together form a contour. For details, see e.g.
S. LANG, *Complex Analysis*, 4th ed., Springer, 1999, IV.2.

5. *The General Form of Cauchy's Theorem.*

For γ a closed path in D, the following are equivalent:
(i) $\int_\gamma f = 0$ for all $f \in H(D)$ (i.e., for all f holomorphic in D);
(ii) for all $f \in H(D)$ and for all z in D but not on γ,

$$W(\gamma, z)f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)} dz,$$

where $W(\gamma, z)$ is the winding number of γ about z;
(iii) The interior $I(\gamma) \subset D$.
See e.g.
R. REMMERT, *Theory of complex functions*, Springer, 1989, Ch. 9 §5.

For more on homotopy and homology, see a course on Algebraic Topology.

NHB