M2PM2 EXAM 2008

1. Let G be the set consisting of the following twelve permutations in S_6:

$$e,\ (1\,2\,3)\,(4\,5\,6),\ (1\,4\,6)\,(2\,5\,3),\ (1\,6\,3)\,(2\,4\,5),\ (1\,4\,2)\,(3\,6\,5),$$

$$(1\,3\,2)\,(4\,6\,5),\ (1\,6\,4)\,(2\,3\,5),\ (1\,3\,6)\,(2\,5\,4),\ (1\,2\,4)\,(3\,5\,6),$$

$$(1\,5)\,(2\,6),\ (1\,5)\,(3\,4),\ (2\,6)\,(3\,4).$$

You are given that G is a subgroup of S_6. Answer the following questions about the group G, giving brief justifications for your answers.

(i) Is G abelian ?

(ii) Find a subgroup of size 3 in G.

(iii) Find a subgroup of size 4 in G.

(iv) Is your subgroup of size 4 normal in G ?

(v) Does G have a subgroup of size 6 ?

(vi) You are given that G is isomorphic to one of the following groups:

$$C_{12},\ C_2 \times C_2 \times C_3,\ D_{12},\ A_4,\ S_3 \times C_2.$$

Which one of these is G isomorphic to ?

2. Let G and H be groups. Define what is meant by a homomorphism $\phi : G \to H$. What does it mean to say that ϕ is surjective ?

(a) Suppose G and H are finite groups and there is a surjective homomorphism $\phi : G \to H$. Stating (but not proving) any standard results you need, prove that $|H|$ divides $|G|$.

(b) For each of the following pairs G, H of groups, say whether or not there exists a surjective homomorphism $\phi : G \to H$, and justify your answer:

(i) $G = D_{10},\ H = C_2$

(ii) $G = D_{10},\ H = C_5$

(iii) $G = S_4,\ H = C_2$

(iv) $G = S_4,\ H = C_6$

(v) $G = S_4,\ H = D_6$.

3. (a) State the Cayley-Hamilton theorem.

(b) Let A be an $n \times n$ matrix over $\mathbb{C}$ such that the only eigenvalue of A is 0. Prove that $A^n = 0$.

(c) Let V be the vector space over $\mathbb{R}$ consisting of all polynomials in x of degree at most 3. Let $T : V \to V$ be the linear transformation defined by

$$T(p(x)) = p(x) + (x+1)p'(x) - x^2 p''(x)$$

for all $p(x) \in V$. (As usual, $p'(x)$ and $p''(x)$ denote the first and second derivatives of $p(x)$.)

(i) Find the eigenvalues of T.

(ii) Is there a basis of V consisting of eigenvectors of T ?

(iii) Find a polynomial f such that $T^{-1} = f(T)$.

4. Let A and B be $n \times n$ matrices, with $n \geq 2$. What does it mean to say that A is similar to B ?

(a) If A is similar to B, and $p(x)$ is any polynomial, prove that $p(A)$ is similar to $p(B)$.

(b) Say which of the following quantities are invariant under similarity (i.e. are the same for any matrix which is similar to A), justifying your answer briefly:

(i) $\det(A^2 - A)$

(ii) $\mathrm{rank}(A + A^T)$

(iii) $\mathrm{trace}(A + A^T)$.

(c) Find the Jordan Canonical Form of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(d) Let

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Is B similar to the matrix A in part (c) ?