M2PM2 Algebra II Problem Sheet 9

Assessed work: there will be a test at the class on Tuesday December 16, 4.00. The test will consist of one surprise question and two of the starred questions from this sheet.

1. Suppose that A and B are $n \times n$ matrices which are similar.

(i) Prove that A^2 is similar to B^2.

(ii) Prove that for any polynomial $f(x)$, the matrices $f(A)$ and $f(B)$ are similar.

2*. Let A be an arbitrary $n \times n$ matrix. Which of the following quantities are invariant under similarity (i.e. are the same for any matrix which is similar to A) ? Give brief justifications for your answers.

(i) $\mathrm{rank}(A^3 - I)$

(ii) $\mathrm{trace}(A + A^5)$

(iii) $c_1(A)$, the sum of the entries in the first column of A

(iv) $\mathrm{rank}(A - A^T)$

(v) $\mathrm{trace}(2A - A^T)$.

3. Suppose that λ is an eigenvalue of a block-diagonal matrix $A = A_1 \oplus \cdots \oplus A_k$. Prove that the geometric multiplicity of λ for A is equal to the sum of its geometric multiplicities for each A_i. (In other words prove that $\dim E_\lambda(A) = \sum_1^k \dim E_\lambda(A_i)$, where $E_\lambda(A)$ and $E_\lambda(A_i)$ are the λ-eigenspaces of A and A_i.)

4*. (i) Write down all the possible Jordan Canonical Forms having characteristic polynomial $x(x + 1 + i)^2(x - 3)^3$.

(ii) Calculate the number of non-similar Jordan Canonical Forms having characteristic polynomial $x^3(x - 1)^6$.

5*. Find the JCFs of the following matrices:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & i & 2 \\ 0 & 0 & 0 & 0 & 0 & i \end{pmatrix}$$

6. Let $J_n(\lambda)$ be a Jordan block. Prove that the matrix $J = J_n(\lambda) - \lambda I$ is similar to its transpose. (*Hint (if needed): consider the linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(v) = Jv$, and try to find bases E, F such that $[T]_E = J$, $[T]_F = J^T$.*)

Deduce that $J_n(\lambda)$ is similar to its transpose.

7. Using Q7 and the JCF theorem, prove that every square matrix over $\mathbb{C}$ is similar to its transpose.

8*. If A is an $n \times n$ matrix, a square root of A is defined to be an $n \times n$ matrix B such that $B^2 = A$.

(i) Give an example of a matrix that has no square root.

(ii) Using the JCF theorem, or otherwise, prove that every invertible matrix A over $\mathbb{C}$ has a square root.