M2PM2 Algebra II Problem Sheet 6

Assessed work: there will be a test at the class on Tuesday November 25, 4.00. At the test, I will choose two of the starred questions and you will be asked to write out solutions to them. No lecture notes, books or other aids allowed.

1*. (a) Let G be a finite group and let H be a subgroup of G of size $\frac{1}{2}|G|$.
Prove that every left coset of H in G is also a right coset.
Hence prove that $H \lhd G$.

(b) Let G be a group, and suppose that G has two normal subgroups M and N such that $M \cap N = \{e\}$. Prove that $mn = nm$ for all $m \in M, n \in N$. (*Hint: consider* $m^{-1}n^{-1}mn$.)

2*. Prove that if G is an abelian group, then every subgroup of G is a normal subgroup.

Disprove the converse of this, by giving an example of a non-abelian group G and showing that every subgroup of G is normal. (*Hint: consider groups of size 8.*)

3. Calculate the determinant of the matrix $\begin{pmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{pmatrix}$.

4*. For a real number α define

$$A(\alpha) = \begin{pmatrix} 1 & \alpha & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 2 & \alpha & 1 & -1 \\ -1 & \alpha & 1 & 1 \end{pmatrix}$$

(a) Find the determinant of $A(\alpha)$.
(b) Find a value α_0 of α such that the system $A(\alpha_0)x = 0$ has a nonzero solution for $x \in \mathbb{R}^4$.
(c) Prove that when $\alpha < \alpha_0$, there is no real 4×4 matrix B such that $B^2 = A(\alpha)$.

5. Let A_n be the $n \times n$ matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ -1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 0 & -1 & 1 & \dots & 1 & 1 & 1 \\ & & & \dots & & & \\ 0 & 0 & 0 & \dots & -1 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix}$$

Prove that $|A_n| = 2^{n-1}$.

6*. Let B_n be the $n \times n$ matrix

$$\begin{pmatrix} -2 & 4 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 4 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}$$

(a) Prove that if $n \geq 4$, then $|B_n| = 8|B_{n-3}|$.
(b) Prove that $|B_n| = 0$ if $n = 3k - 1$ (where k is a positive integer).
(c) Find $|B_n|$ if $n = 3k$ or $3k + 1$.

7*. Let A, B be $n \times n$ matrices.
(a) Prove that if $|A| = 0$ then $|AB| = 0$.
(b) Prove that if $|B| = 0$ then $|AB| = 0$.
Hint: you may NOT assume the result $|AB| = |A|\,|B|$ *from lecs (this qn is supposed to be part of the proof of that result). But you MAY assume the result in lecs that says a matrix is invertible iff it has nonzero determinant.*