M2P2 Algebra II
Solutions to Problem Sheet 9

1. If $B = P^{-1}AP$ then $B^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$ and similarly $f(B) = P^{-1}f(A)P$ for any poly $f(x)$.

2. (i) Yes: if B is similar to A then $B^3 - I$ is similar to $A^3 - I$, so $\text{rank}(B^3 - I) = \text{rank}(A^3 - I)$ by 15.1 of lecs.

(ii) Yes: same proof

(iii) No: eg $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are similar but have different first row sum.

(iv) No: eg let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Then A and B are similar, but $A - A^T = 0$ has rank 2, whereas $B - B^T$ has rank 0.

(v) Yes: this is a cunning trick, as A and A^T have the same diagonal entries, so $\text{trace}(2A - A^T) = \text{trace}(A)$, which is invariant by 15.1 of lecs.

3. This question is fairly easy, but notationally awkward. Say each A_i is $n_i \times n_i$, so A is $n \times n$ where $n = \sum n_i$. Write each column vector in F^n ($F = \mathbb{R}$ or $\mathbb{C}$) in the form $v = (v_1, v_2, \dots, v_k)$, where $v_i \in F^{n_i}$ for all i. Then $Av = (A_1v_1, A_2v_2, \dots, A_kv_k)$. Hence $Av = \lambda v$ if and only if $A_iv_i = \lambda v_i$ for all i.

Let $E_\lambda(A_i)$ be the λ-eigenspace of A_i, and let B_i be a basis of $E_\lambda(A_i)$. Each vector $b \in B_i$ gives a vector $(0, \dots, b, \dots 0)$ in F^n. Let B_i' be the set of such vectors obtained from B_i. By the previous observation, vectors in $E_\lambda(A)$ are of the form $(v_1, v_2, \dots, v_k)$ with $v_i \in E_\lambda(A_i)$. These are linear combinations of the vectors in $\cup B_i'$. Hence $\cup B_i'$ is a basis for $E_\lambda(A)$. So $\dim E_\lambda(A) = \sum |B_i'| = \sum |B_i| = \sum \dim E_\lambda(A_i)$.

4. (i) $J_1(0) \oplus J_1(-1-i)^2 \oplus J_1(3)^3$, $J_1(0) \oplus J_1(-1-i)^2 \oplus J_2(3) \oplus J_1(3)$, $J_1(0) \oplus J_1(-1-i)^2 \oplus J_3(3)$, $J_1(0) \oplus J_2(-1-i) \oplus J_1(3)^3$, $J_1(0) \oplus J_2(-1-i) \oplus J_2(3) \oplus J_1(3)$, $J_1(0) \oplus J_2(-1-i) \oplus J_3(3)$. Phew!

(ii) There are 3 possible JCFs with char poly x^3 ($J_3(0)$, $J_2(0) \oplus J_1(0)$ etc) and 11 with char poly $(x-1)^6$ ($J_6(1)$, $J_5(1) \oplus J_1(1)$ etc). So there are 33 JCFs with char poly $x^3(x-1)^6$.

5. $J_1(1) \oplus J_1(0) \oplus J_1(-1)$, $J_1(3) \oplus J_1(0)^2$, $J_1(-1) \oplus J_2(2)$, $J_4(0) \oplus J_1(0)$, $J_3(-1) \oplus J_1(-1) \oplus J_2(i)$.

6. Let E be the standard basis in order $e_1, \dots, e_n$ and F the standard basis in reverse order $e_n \dots, e_1$. As $Je_n = e_{n-1}$, $Je_{n-1} = e_{n-2}$, etc, the linear transformation $T(v) = Jv$ satisfies $[T]_E = J$, $[T]_F = J^T$. So if P is the change of basis matrix from E to F, $P^{-1}JP = J^T$. Therefore J and J^T are similar.

Finally,

$$P^{-1}J_n(\lambda)P = P^{-1}(J + \lambda I)P = J^T + \lambda I = (J + \lambda I)^T = J_n(\lambda)^T$$

so $J_n(\lambda)$ and $J_n(\lambda)^T$ are similar.

7. Let A be an $n \times n$ matrix over $\mathbb{C}$. By the JCF theorem A is similar to a JCF matrix $J = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$. By Q6, for each i, $\exists P_i$ such that $P_i^{-1} J_{n_i}(\lambda_i) P_i = J_{n_i}(\lambda_i)^T$. If we let P be the block-diagonal matrix $P_1 \oplus \cdots \oplus P_k$, then $P^{-1} = P_1^{-1} \oplus \cdots \oplus P_k^{-1}$ and so

$$P^{-1}JP = P_1^{-1} J_{n_1}(\lambda_1) P_1 \oplus \cdots \oplus P_k^{-1} J_{n_k}(\lambda_k) P_k = J_{n_1}(\lambda_1)^T \oplus \cdots \oplus J_{n_k}(\lambda_k)^T = J^T.$$

So J is similar to J^T, and hence A is similar to J^T, i.e. $\exists Q$ such that $Q^{-1}AQ = J^T$. Taking transposes, $Q^T A^T (Q^{-1})^T = J$. Since $(Q^{-1})^T = (Q^T)^{-1}$ (see the days of M1GLA), this shows A^T is similar to J. So both A and A^T are similar to J, whence A is similar to A^T. Phew!

8. (i) E.g. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(ii) This is a really nice application of the JCF. The answer is yes. Here is a sketch. Since A is similar to a direct sum of Jordan blocks $J_r(\lambda)$ (with $\lambda \neq 0$ as A is invertible), it is enough to show that each such Jordan block $J_r(\lambda)$ has a square root. Let μ be a square root of λ in $\mathbb{C}$. Consider $J_r(\mu) = J + \mu I$ where $J = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & & \cdots \end{pmatrix}$. Then $J_r(\mu)^2 = J^2 + 2\mu J + \mu^2 I$. Argue that the JCF of this matrix is $J_r(\mu^2) = J_r(\lambda)$. Hence $\exists P$ such that $P^{-1} J_r(\mu)^2 P = J_r(\lambda)$, i.e. $(P^{-1} J_r(\mu) P)^2 = J_r(\lambda)$. Hence $J_r(\lambda)$ has a square root, as desired.