M2PM2 Algebra II
Solutions to Problem Sheet 8

1. (a)(i) Char. poly is $(x+1)^2(x-2)$, so evalues are $-1, 2$ with alg multiplicities 2,1 respectively. Geom multiplicity of the evalue -1 is dimension of the -1 eigenspace, which is 1; geom mult of 2 is also 1. Since geom mult of -1 is less than alg mult, there is no basis of evectors.

(ii) T sends $1 \to 0$, $x \to 3x$, $x^2 \to x + 6x^2$, so matrix of T wrt basis $1, x, x^2$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{pmatrix}$. This has distinct evalues 0,3,6, all with alg and geom mult 1, and there is a basis of evectors.

(b) The char poly is $(x+1)^2(x-1)$, so A is diagonalisable iff the -1 eigenspace has dimension 2. This eigenspace consists of solutions to the system $\begin{pmatrix} 0 & a & b \\ 0 & 2 & c \\ 0 & 0 & 0 \end{pmatrix} x = 0$, so it is 2-dimensional iff $ac - 2b = 0$.

2. If A, B are similar then $\exists P$ such that $B = P^{-1}AP$, so the char poly of B is

$$det(xI - P^{-1}AP) = det(P^{-1}(xI - A)P) = det(xI - A)$$

(using result in lecs saying $det(P^{-1}XP) = detX$), which is the char poly of A.

3. We are given that $\exists P$ such that $P^{-1}AP = D = \text{diag}(\lambda_1, \ldots, \lambda_n)$, the diagonal matrix with diagonal entries λ_i. By Q2, A has the same char poly as D, namely $p(x) = \prod(x - \lambda_i)$. Clearly $p(D) = \text{diag}(p(\lambda_1), \ldots, p(\lambda_n)) = 0$. Since $A = PDP^{-1}$, $A^2 = PD^2P^{-1}$ and so on, we see that $p(A) = Pp(D)P^{-1} = 0$.

4. By induction on n. The char poly is

$$p(x) = det \begin{pmatrix} x & 0 & 0 & \cdots & 0 & a_0 \\ -1 & x & 0 & \cdots & 0 & a_1 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & -1 & x + a_{n-1} \end{pmatrix}$$

Expand by the first row. By induction the det of the 11-minor is $x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$, so we get

$$p(x) = x\,(x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1) + (-1)^{n-1}a_0.(-1)^{n-1} = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.$$

Hence the result by induction.

5. (a) $\begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 7 \end{pmatrix}$ works (by Q4)

(b) If we find A with char poly $x^3 - 2x^2 - 1$ then A will satisfy the desired equation by Cayley-Hamilton. So take $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

(c) Multiplying through by B, the eqn is $B^4 + B - I = 0$. So finding B with char poly $x^4 + x - 1$ will do. use Q4 to do this.

(d) By Q4 the 2×2 matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ satisfies $A^2 + A + I = 0$. So take $C = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$.

(e) Use Q4 to get a non-identity $n \times n$ matrix with char poly $x^n - 1$.

6. Since the only evalues of A are 0 and 1, these are the only roots of the char poly, which must therefore be $x^k(x-1)^{n-k}$ for some k. Hence by Cayley-Hamilton, $A^k(A-I)^{n-k} = 0$, and so $A^n(A-I)^n = 0$.