M2P2 Algebra II
Solutions to Problem Sheet 7

1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. First bit done in lecs. To show $\sim$ an equiv rel: obviously $A \sim A$; if $A \sim B$ then $B = E_1 \ldots E_k A$, hence $A = E_k^{-1} \ldots E_1^{-1} B$, so $B \sim A$ as all E_i^{-1} are elementary; and if $A \sim B$ and $B \sim C$, then $B = E_1 \ldots E_k A$ and $C = F_1 \ldots F_l B$ with all E_i, F_i elementary, so $C = F_1 \ldots F_l E_1 \ldots E_k A$, hence $A \sim C$.

3. (i) det is 0

(ii) Matrix of T w.r.t usual basis $1, x, x^2, x^3$ is triangular with diagonal entries all 1, so has det 1.

(iii) Matrix of T w.r.t. basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$, which has det 1.

4. (i) Calculate that $q(T) = I$. This has det 1.

(ii) T sends $1 \to 1$, $x \to x$, $x^2 \to x^2 + 4x - 1$, $x^3 \to x^3 + 9x - 2$, so matrix of T w.r.t. basis $1, x, x^2, x^3$ is

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This has only evalue 1, and basis for 1-eigenspace of T is $1, x$. There is no basis of evectors.

5. Consider a term $sgn(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)}$ in $det(A)$, where $\pi \in S_n$. Because of the $t \times s$ zero matrix in the bottom left of A, for this term to be non-zero, it is necessary that π sends $\{1, \ldots s\} \to \{1, \ldots s\}$ and $\{s+1, \ldots s+t\} \to \{s+1, \ldots s+t\}$. We can write such a π as a product $\pi_1 \pi_2$, where π_1 is a permutation of $\{1, \ldots s\}$ and π_2 is a permutation of $\{s+1, \ldots s+t\}$. Also $sgn(\pi) = sgn(\pi_1)\, sgn(\pi_2)$. Hence

$$det(A) = \sum_{\pi_1, \pi_2} sgn(\pi_1) sgn(\pi_2) b_{1,\pi_1(1)} \cdots b_{s,\pi_1(s)}\, d_{s+1,\pi_2(s+1)} \cdots d_{s+t,\pi_2(s+t)} =$$

$$\sum_{\pi_1} sgn(\pi_1) b_{1,\pi_1(1)} \cdots b_{s,\pi_1(s)} \sum_{\pi_2} sgn(\pi_2) d_{s+1,\pi_2(s+1)} \cdots d_{s+t,\pi_2(s+t)} = det(B)\, det(D).$$

6. Define $A \sim B$ if $\exists P$ such that $B = P^{-1} A P$.

Then $A \sim A$ as $A = I^{-1} A I$.

And $A \sim B \Rightarrow B = P^{-1} A P \Rightarrow A = P B P^{-1} \Rightarrow B \sim A$.

Finally $A \sim B$, $B \sim C \Rightarrow B = P^{-1}AP$, $C = Q^{-1}BQ \Rightarrow C = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ) \Rightarrow A \sim C$.

Hence $\sim$ is an equivalence relation.

7. Routine: $P = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$, $Q = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$, $[v]_E = (a,b)^T$, $[v]_F = (-5a + 2b, 3a - b)^T$, $[T]_E = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}$, $[T]_F = \begin{pmatrix} -30 & -48 \\ 18 & 29 \end{pmatrix}$.

8. (i) Closure: $S, T \in GL(V)$ implies that ST is a linear trans, and is invertible as $(ST)^{-1} = T^{-1}S^1$, so $ST \in GL(V)$.

Assoc: follows from assoc of composition

Identity: is identity map $I(v) = v \forall v \in V$.

Inverse: exists by defn.

Hence $GL(V)$ is a group.

(ii) Fix a basis B of V. Then the map $T \to [T]_B$ is an isomorphism from $GL(V) \to GL(n, \mathbb{R})$.