M2PM2 Algebra II
Solutions to Problem Sheet 6

1. (a) Let $x \in G - H$. Then $Hx \neq H$, so as $|H| = \frac{1}{2}|G|$, we have $G = H \cup Hx$. Similarly $xH \neq H$ so $G = H \cup xH$. Therefore $Hx = G - H = xH$. So every left coset is a right coset.

Let $x \in G$. If $x \in H$ then obviously $x^{-1}Hx = H$. And if $x \notin H$ then by the above, $xH = Hx$, so $H = x^{-1}Hx$. Hence $H \lhd G$.

(b) Consider $x = m^{-1}n^{-1}mn$. As $M \lhd G$, $n^{-1}Mn = M$ and so $n^{-1}mn = m' \in M$, therefore $x = m^{-1}m' \in M$. Similarly $N \lhd G$, so $m^{-1}n^{-1}m = n' \in N$ and so $x = n'n \in N$. We conclude that $x \in M \cap N$. Since we are given that $M \cap N = \{e\}$ it follows that $x = e$, so $m^{-1}n^{-1}mn = e$, which implies that $mn = nm$.

2. Let H be a subgroup of the abelian group G. For $g \in G$ and $h \in H$ we have $gh = hg$, hence $g^{-1}hg = h$, and so $g^{-1}Hg = H$. Therefore $H \lhd G$.

Consider the quaternion group Q_8 defined in Sheet 3, Q5. Check that Q_8 has one element of order 2 (namely $A^2 = -I$) and all other non-identity elements have order 4. Hence, apart form the obvious subgroups $\{e\}$ and Q_8 itself, Q_8 has one subgroup of size 2, namely $\langle A^2 \rangle$, and all other subgroups have size 4. The subgroup $\langle A^2 \rangle$ is normal in Q_8 as $g^{-1}A^2g = A^2$ for all $g \in Q_8$. And the subgroups of size 4 are all normal in Q_8 by Question 1(a).

3. -102 (I think!)

4. (a) $|A(\alpha)| = \alpha - 1$

(b) $\alpha_0 = 1$ (using result from lecs that system $Ax = 0$ has a nonzero soln for x iff $|A| = 0$).

(c) For $\alpha < 1$, $|A(\alpha)| < 0$. If $B^2 = A(\alpha)$ then by the multiplicativity of det, $|B|^2 = |A(\alpha)| < 0$, which is impossible if B is real.

5. Expanding by 1st col, get $|A_n| = |A_{n-1}| + |A_{n-1}| = 2|A_{n-1}|$. So

$$|A_n| = 2|A_{n-1}| = 2 \cdot 2|A_{n-2}| = \cdots = 2^{n-2}|A_2| = 2^{n-1}.$$

6. (a) Expanding by 1st col, get

$$|B_n| = -2|B_{n-1}| - 4\,det \begin{pmatrix} 4 & 0 & 0 & \dots \\ 1 & 2 & -4 & \dots \\ & & & \dots \end{pmatrix} = -2|B_{n-1}| - 4|B_{n-2}|.$$

Substitute for $|B_{n-1}|$ in this, using the same formula ($|B_{n-1}| = -2|B_{n-2}| - 4|B_{n-3}|$). This gives

$$|B_n| = 8|B_{n-3}|.$$

(b) When $n = 3k - 1$ this shows that

$$|B_n| = 8|B_{n-3}| = 8^2|B_{n-6}| = \cdots = 8^{k-1}|B_2| = 0.$$

(c) When $n = 3k$, we similarly get $|B_n| = 8^{k-1}|A_3| = 8^k$. And when $n = 3k+1$, we get $|B_n| = 8^k|A_1| = -2^{3k+1}$.

7. (a) Suppose $|A| = 0$. Then A is not invertible (result 11.6 in lecs). It follows that AB is also not invertible (if it were, say the inverse was C, we'd have $ABC = I$, so BC would be the inverse of A, contradiction). Hence $|AB| = 0$, again by 11.6 of lecs.

(b) Similar: suppose $|B| = 0$. Then B is not invertible. It follows that AB is also not invertible (if it were, say the inverse was C, we'd have $CAB = I$, so CA would be the inverse of B, contradiction). Hence $|AB| = 0$.