M2PM2 Algebra II
Solutions to Problem Sheet 5

1. There are such homoms for $r = 1$ ($\phi(g) = 1$ for all $g \in S_n$) and for $r = 2$ ($\phi(g) = sgn(g)$ for all $g \in S_n$).

We now show there is no such homom if $r > 2$. For suppose $\phi : S_n \to C_r$ is an onto homom. Then $\phi(ij)$ has order 1 or 2, so is ± 1. As every perm is a product of 2-cycles (see lecs), this forces $\phi(g) = \pm 1$ for all $g \in S_n$. Therefore $r \leq 2$.

2. (a) For $x, y \in G$, $(Nx)(Ny) = Nxy = Nyx = (Ny)(Nx)$. Hence G/N is abelian.

(b) $G = S_3, N = A_3$. Then N is abelian, as is $G/N \cong C_2$.

(c) Let $G = D_8$, $N = \{e, \rho^2, \sigma, \rho^2\sigma\}$ and $M = \langle\sigma\rangle$. Then $N \triangleleft G$ as shown in an example in lecs, and $M \triangleleft N$ as N is abelian. But M is not normal in G.

3. (a) G is cyclic, hence abelian, so $N \triangleleft G$ as shown in lecs. If $G = \langle x \rangle$, then every coset in the factor group G/N is of the form $Nx^r = (Nx)^r$, so G/N is generated by Nx, so is cyclic.

(b) $G = D_8$: possible H have size dividing 8, so 1,2,4 or 8. Examples in lecs show that C_1, C_2 and D_8 are possible H. So the only question is which H of size 4 are possible. Q6 of Sheet 4 shows only $C_2 \times C_2$ is poss. So the list of H is C_1, C_2, $C_2 \times C_2$ and D_8

$G = D_{12}$: by lecs the possible H are the factor groups G/N for $N \triangleleft G$. As above the only question is which H of size 4 or 6 are poss. For $|H| = 4$ we need $|N| = 3$, and the only normal subgroup of size 3 is $N = \langle\rho^2\rangle$, for which $G/N \cong C_2 \times C_2$ by lecs. For $|H| = 6$ we need $|N| = 2$, and the only normal subgroup of size 2 is $N = \langle\rho^3\rangle$, for which $G/N \cong D_6$ by lecs. The poss H are C_1, C_2, $C_2 \times C_2$, D_6, D_{12}.

$G = (\mathbb{Z}, +)$: the only (automatically normal) subgroups of $\mathbb{Z}$ are of the form $k\mathbb{Z} = \{kn : n \in \mathbb{Z}\}$. (Proof: let K be a subgroup of $\mathbb{Z}$, and assume $K \neq \{0\}$. Then K contains some positive integers. Let k be the smallest positive integer in K. We claim that $K = k\mathbb{Z}$. To see this, let $n \in K$. Then $n = qk + r$, where $q, r \in \mathbb{Z}$ and $0 \leq r < k$. Then $n \in K$ and $qk \in K$, hence $n - qk = r \in K$. By the minimal choice of k, we must have $r = 0$. So $n = qk$. This shows that $K = k\mathbb{Z}$.)

Therefore the possibilities for H are the factor groups $\mathbb{Z}/k\mathbb{Z}$ ($k \in \mathbb{N}$). These are cyclic groups (by part (a)) and are isomorphic to C_k.

4. The total number of colourings is $\binom{8}{3} \times \binom{5}{3} = 560$. The number of distinguishable colourings is the number of robits of $G = D_{16}$ on the set of colourings. The number of fixed colourings of each of the elements of G are as follows:

$g:$	e	ρ^i	$\rho^{2i}\sigma\ (i = 0, 1, 2, 3)$	$\rho^{2i+1}\sigma\ (i = 0, 1, 2, 3)$
$fix(g):$	560	0	0	12

(where $\rho^{2i}\sigma$ are the reflections in an axis bisecting two opposite sides, and $\rho^{2i+1}\sigma$ are the other reflections). Hence by Burnside's lemma, number of orbits = average number of fixed points which is $\frac{1}{16}(560 + 48) = 38$.

5. If no colour used more than once, $fix(g) = 0$ for all $g \in D_8 - \{e\}$, so no. of distinguishable colourings is $\frac{1}{8}(fix(e)) = (6.5.4.3)/8 = 45$. If colours can be used more than once, check that fixed point numbers are:

$$\begin{array}{ccccccccc} g: & e & \rho & \rho^2 & \rho^3 & \sigma & \rho\sigma & \rho^2\sigma & \rho^3\sigma \\ fix(g): & 6^4 & 6 & 6^2 & 6 & 6^3 & 6^2 & 6^3 & 6^2 \end{array}$$

So no. of dist. colourings $= \frac{1}{8}(6^4 + \ldots + 6^2) = 231$.

6. (a) Group is rotation group of tetra which is $G \cong A_4$. Check that $fix(e) = 3^4$, while $fix(g) = 3^2$ for all $g \neq e$. hence number of dist. colourings is $\frac{1}{12}(3^4+11\cdot 3^2) = 15$.

(b) Check that $fix(g) = 4^4$ if $g = e$, 4^2 if g is a rotation of order 3, and 4^3 if g is a rotation of order 2. So no. of dist. colourings is $\frac{1}{12}(4^4 + 8.4^2 + 3.4^3) = 48$.

7. Very briefly: if you consider each rotation in the symmetry group of the cube as a permutation of the 4 long diagonals, you get the eight 3-cycles by rotating about axes which are long diagonals; you get the six 2-cycles by rotating about axes through the mid-points of diagonally opposite sides; and you get the 4-cycles and $(2, 2)$-perms by rotating about axes through the mid points of opposite faces.

8. (a) This is the number of orbits of the rotation group R of the cube in Q6. Check that $fix(e) = 6!$, while $fix(g) = 0$ for $g \neq e$. Hence number of orbits $= \frac{1}{24}(6!) = 30$.

(b) Here the number of colourings is $\binom{6}{2}.\binom{4}{2}.2.1$ (choose 2 faces for the 4's, then 2 faces for the 3's, then 1 face for the 2). So $fix(e) = 180$. Of the rotations, only the ones giving $(2, 2)$-perms as in Q6 fix any colourings, and these fix 4. So number of orbits is $\frac{1}{24}(180 + 3 \cdot 4) = 8$.