M2PM2 Algebra II
Solutions to Problem Sheet 4

1. Yep, image C_4, kernel C_3; nope; yep, image $\mathbb{Z}_n$, kernel $\langle n\rangle = \{kn : k \in \mathbb{Z}\}$; yep, image $\mathbb{R}_{>0}$, kernel $\{\pm 1\}$; nope.

2. (a) Suppose G is abelian. Then for $x, y \in G$, $\phi(xy) = (xy)^2 = xyxy = xxyy = x^2y^2 = \phi(x)\phi(y)$.

(b) Suppose G is non-abelian. Then there are elements $a, b \in G$ such that $ba \neq ab$. Then $(ab)^2 \neq a^2b^2$ (otherwise $abab = aabb$, which implies $ba = ab$). Hence $\phi(ab) \neq \phi(a)\phi(b)$, so ϕ is not a homomorphism.

3. Let $g \in G$ and $x \in M \cap N$. Then $g^{-1}xg \in M$ (as $M \lhd G$) and $g^{-1}xg \in N$ (as $N \lhd G$), hence $g^{-1}xg \in M \cap N$. Thus $g^{-1}(M \cap N)g \subseteq M \cap N$. As in lecs, this implies $M \cap N \lhd G$.

4. (a) We prove this by induction on k. It's true for $k = 1$. Assume for $k - 1$, so $\sigma\rho^{k-1} = \rho^{-(k-1)}\sigma$. Then

$$\sigma\rho^k = \sigma\rho^{k-1}\rho = \rho^{-(k-1)}\sigma\rho = \rho^{-(k-1)}\rho^{-1}\sigma = \rho^{-k}\sigma,$$

hence the result by induction.

(b) Let $x = \rho^{ri} \in \langle\rho^r\rangle$. Then

$$\begin{aligned}&\rho^{-j}x\rho^j = \rho^{-j+ri+j} = x,\\ &(\rho^j\sigma)^{-1}x(\rho^j\sigma) = (\sigma\rho^{-j})\rho^{ri}(\rho^j\sigma) = \sigma\rho^{ri}\sigma = \rho^{-ri}\sigma\sigma = \rho^{-ri} = x^{-1}.\end{aligned}$$

Hence $g^{-1}xg \in \langle\rho^r\rangle$ for all $g \in D_{2n}$, so $\langle\rho^r\rangle \lhd D_{2n}$.

(c) $\rho^{-1}(\rho^r\sigma)\rho = \rho^{-1}\rho^r\rho^{-1}\sigma = \rho^{r-2}\sigma \notin \langle\rho^r\sigma\rangle$ (using $n \geq 3$ here), so $\langle\rho^r\sigma\rangle$ is not normal in D_{2n}.

5. (a) Let H be a subgroup of D_{2p}, and assume $H \neq \{e\}$ or D_{2p}. By Lagrange, H has size 2 or p, so H is cyclic. If $|H| = 2$ then H is generated by a reflection σ: as $\rho^{-1}\sigma\rho \neq e$ or σ, this is not normal in D_{2p}. If $|H| = p$ then $H = \langle\rho\rangle$, which is normal (lecs).

Therefore the normal subgroups of D_{2p} just $\{e\}, D_{2p}$ and $\langle\rho\rangle$.

(b) By lecs, the groups H for which there is a homom from D_{2p} onto H are the groups D_{2p}/N, where $N \lhd D_{2p}$. Hence the groups H are C_1, D_{2p} and C_2.

6. (i) Yes, homom $\phi(x) = x^2$.

(ii) No, 5 does not divide 12. (As $|G|/|\ker\phi| = |\mathrm{Im}\ \phi|$, $|\mathrm{Im}\ \phi|$ divides $|G|$.)

(iii) No: suppose ϕ is a homom from D_8 onto C_4. Then $\ker\phi$ has size 2. Let $K = \ker\phi$. As $K \lhd D_8$, K is not generated by a reflection, hence $K = \langle\rho^2\rangle$. By the Isomorphism Theorem in lecs, $D_8/K \cong \mathrm{Im}\ \phi = C_4$. But $D_8/\langle\rho^2\rangle \cong C_2 \times C_2$ (because each of the 4 right cosets K, $K\rho$, $K\sigma$, $K\rho\sigma$ has order 2), which is a contradiction.

(iv) Yes: let $N = \langle\rho^2\rangle \lhd D_8$. As in the previous part, $D_8/N \cong C_2 \times C_2$. Hence the map $x \to Nx$ is a homom from D_8 onto $C_2 \times C_2$.

7. (i) Draw up a mult table to show V is closed. It contains e and each of its elements is self-inverse, so V is a subgroup.

(ii) For $v \in V$, we have $v^2 = e$; so for $g \in S_4$,

$$(g^{-1}vg)^2 = g^{-1}vgg^{-1}vg = g^{-1}v^2g = g^{-1}eg = e.$$

Therefore if $v \neq e$, $g^{-1}vg$ has order 2. It is also an even perm. All even perms of order 2 lie in V, hence $g^{-1}Vg \subseteq V$, hence $V \triangleleft S_4$.

(iii) Let $x = V(123), y = V(12) \in S_4/V$. Check that $o(x) = 3, o(y) = 2$ and $yx = x^{-1}y$. Hence $S_4/V \cong D_6$ as in lecs.