M2P2 Algebra II   Solutions to Sheet 3

1. Cycle-shapes $e, (3), (2,2), (5)$, numbers 1, 20, 15, 24 respectively.

2. Use Fund Theorem of Abelian Groups to get:

(a) One: C_{21}. (Point is that $C_3 \times C_7 \cong C_{21}$)

(b) Two: C_{12}, $C_2 \times C_6$. (Notice $C_2 \times C_2 \times C_3 \cong C_2 \times C_6$)

(c) Five: $(C_3)^4$, $(C_3)^2 \times C_9$, $C_3 \times C_{27}$, $(C_9)^2$, C_{81}. Check no two of these are isomorphic by showing they have different numbers of elements of some order.

Marks: 1,1,2

3. (a) 12 (cycle-shape $(4,3)$)

(b) Suppose S_7 has a subgroup isomorphic to D_{2n}. Then S_7 has an element of order n since D_{2n} does. The orders greater than 7 of elements of S_7 are 10 and 12 (cycle-shapes $(5,2)$ and $(4,3)$).

(c) Yes: let $x = (1234)(567)$ and $y = (13)(56)$, and check that

$$x^{12} = e,\ y^2 = e,\ yx = x^{-1}y.$$

Let $G = \{e, x, \ldots, x^{11}, y, xy, \ldots, x^{11}y\}$. Then G is a subgroup of S_5 (closure and inverses can be proved using the above equations). As we saw in examples in lecs, the above equations determine the mult table of G. As they are the same as the equations for D_{24}, conclude that $G \cong D_{24}$.

Marks: 1,1,3

4. (i) $C_2 \times \cdots \times C_2$ (n factors)

(ii) $D_8 \times D_8$ (many other possibs)

(iii) $\mathbb{Z} \times D_6$, where $\mathbb{Z}$ is the integers under addition. The abelian subgroup H is $\mathbb{Z} \times \langle \rho \rangle$, where ρ is a rotation of order 3 in D_6. (Many other possibs)

Marks: 1,2,2

5. (a) Easy

(b) By (a) we will get all the matrices $A^r B^s$ if we take $0 \le r \le 3$ and $0 \le s \le 1$ (note the upper limit 1 rather than 3 for s, since we can replace B^2 by A^2). These matrices are

$$\pm I,\ \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \pm \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

(c) We check the 3 subgroup properties:

(1) $I \in Q_8$

(2) Closure: using the equation $BA = A^3B$, we see that any product $(A^r B^s)(A^t B^u)$ is again of the form $A^m B^n$, so is in Q_8.

(3) Inverses: the inverse of A^rB^s is $B^{-s}A^{-r}$, and using the equation $BA = A^3B$, we see this is again of the form A^mB^n, so is in Q_8.

Hence Q_8 is a subgroup of $GL(2, \mathbb{C})$.

(d) Check from the list of matrices in (b) that Q_8 has only 1 element of order 2 (namely $-I$). Since D_8 has 5 elements of order 2, it follows that $Q_8 \not\cong D_8$.

6. (a) Let G be a non-abelian group with $|G| = 8$. The elements of G have order 1,2,4 or 8 by Lagrange. Now G has no element of order 8 (otherwise $G \cong C_8$ which is abelian), and not every element x satisfies $x^2 = e$ (otherwise G would be abelian by Sheet 2, Q4). Hence G has an element x of order 4.

(b) We are given that $y \neq x^2$, and also $y \neq x$ or x^{-1} as these have order 4. So $y \in G - \langle x \rangle$ and

$$G = \langle x \rangle \cup \langle x \rangle y = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}.$$

Consider the product yx. It is clearly not e, x, x^2, x^3 or xy (the last would force G to be abelian). So $yx = x^2y$ or x^3y. If $yx = x^2y$ then there are lots of ways of fiddling around to get a contradiction. Here's one:

$$yx = x^2y \Rightarrow x^2 = yxy^{-1} \Rightarrow e = (x^2)^2 = (yxy^{-1})(yxy^{-1} = yx^2y^{-1} \Rightarrow x^2 = e$$

which is a contradiction.

Hence $yx = x^3y$. Now we have the equations

$$x^4 = e,\ y^2 = e,\ yx = x^3y.$$

These equations determine the mult table of G, and as they are also the equations determining the mult table of D_8, it follows that $G \cong D_8$.

Marks: 2,4

7. By Q6(a), G has an element x of order 4. Pick $y \in G - \langle x \rangle$. Then

$$G = \langle x \rangle \cup \langle x \rangle y = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}.$$

Consider the product yx. Show exactly as in Q6(b) that $yx = x^3y$.

If y has order 2 then $G \cong D_8$ by Q6(b). The only other possibility is that y has order 4, so assume this now. Consider y^2. It cannot be equal to e, x or x^3 (the latter two have order 4). It cannot be y, xy, x^2y, x^3y as $y \notin \langle x \rangle$. So $y^2 = x^2$. We now have the equations

$$x^4 = e,\ x^2 = y^2,\ yx = x^3y.$$

These equations determine the mult table of G, and as they are also the equations determining the mult table of Q_8, it follows that $G \cong Q_8$.

8. (a) By cor. to Lagrange, non-identity elements have order 3 or 9. There is no element of order 9 (otherwise G would be cyclic, hence abelian).

(b) Let x be a non-identity element of G, and let $y \in G - \langle x \rangle$. By (a), x, y both have order 3. If $x^i y^j = x^k y^l$ for some $0 \leq i, j, k, l \leq 2$, then $i = k, j = l$ (otherwise y would be in $\langle x \rangle$). Hence $x^i y^j$ $(0 \leq i, j \leq 2)$ are 9 different elements of G, so are all the elements of G.

(c) Consider yx. By (b) it is equal to $x^i y^j$ for some $0 \leq i, j \leq 2$. It is clearly not e, x, x^2, y or y^2, so it must be one of xy, x^2y, xy^2, x^2y^2.

If $yx = x^2y$ then $(yx)^2 = yxyx = x^2yx^2y = x^2(yx)xy = x^2x^2yxy = x^2x^2x^2yy = y^2$, so $(yx)^3 = y^2yx = x$. But by (a), yx has order 3, so $(yx)^3 = e$, a contradiction. We get similar contradictions if $yx = xy^2$ or x^2y^2. Therefore $yx = xy$.

(d) Since $yx = xy$ we see that $(x^iy^j)(x^ky^l) = (x^ky^l)(x^iy^j)$ for all i, j, k, l. Hence G is abelian, a contradiction (we assumed in (a) that G was non-abelian).

9. By Q7 groups of size 9 are abelian. By Fund Theorem of Abelian Groups, the possibilities are C_9 and $C_3 \times C_3$. (These are not isomorphic, as the latter has no element of order 9.)