M2PM2 Algebra II **Solutions to Sheet 2**

1. We must show three things: (i) $G \cong G$, (ii) $G \cong H \Rightarrow H \cong G$, and (iii) $G \cong H$, $H \cong K \Rightarrow G \cong K$.

For (i), observe that the identity function $f(x) = x$ ($x \in G$) is an isomorphism from G to G.

For (ii), let $\phi : G \to H$ be an isom. We claim ϕ^{-1} is an isomorphism $H \to G$. It is a bijection (1st yr M1F). And for $a, b \in H$, we have $a = \phi(c), b = \phi(d)$ for some $c, d \in G$, hence $\phi^{-1}(ab) = \phi^{-1}(\phi(c)\phi(d)) = \phi^{-1}(\phi(cd)) = cd = \phi^{-1}(a)\phi^{-1}(b)$. Hence $\phi^{-1} : H \to G$ is an isom, so $H \cong G$.

For (iii), let $\phi : G \to H$ and $\psi : H \to K$ be isoms. Then $\psi \circ \phi : G \to K$ is a bijection (M1F again), and is an isomorphism since for all $x, y \in G$,

$$\psi \circ \phi(xy) = \psi(\phi(xy)) = \psi(\phi(x)\phi(y)) = \psi(\phi(x))\, \psi(\phi(y)) = \psi \circ \phi(x)\, \psi \circ \phi(y).$$

Hence $G \cong K$.

2. $\phi(e) = e$ by lecs, so $e = \phi(gg^{-1}) = \phi(g)\phi(g^{-1})$ hence $\phi(g^{-1}) = \phi(g)^{-1}$.

3. Call these groups $G_1, \ldots, G_5$ in the order they are listed. Then $G_2 = (\mathbb{Z}, +) \cong \langle \pi \rangle = G_4$ as they are both infinite cyclic. Also $G_3 = (\mathbb{Q}^*, \times) \cong G_5$, an isomorphism being $a \to a-1$. There are no further isomorphisms between these groups.

4. (a) D_{120} has elements of order 60, whereas S_5 does not, so $S_5 \not\cong D_{120}$ by Prop 3.1 of lecs. And C_{120} is not isomorphic to either of these groups as it is abelian and the others are not.

(b) Isomorphism $\phi : D_6 \to S_3$ is given by sending each element of D_6 to the corresponding permutation of the corners of the triangle.

(c) Isomorphism $x \to e^x$.

(d) One subgroup of size 4 is $\langle \rho \rangle$, the subgroup consisting of all rotations. Another is the subgroup consisting of the symmetries $e, \rho^2, \sigma, \sigma\rho^2$. These subgroups are not isomorphic as one is cyclic and the other is not.

5. (a) Let $x, y \in G$. Then $x^2 = y^2 = (xy)^2 = e$. So $e = xxyy = xyxy$. Multiply on left by x^{-1} and on right by y^{-1}, to get $xy = yx$. Hence G is abelian.

(b) Suppose $|G| > 2$. Pick non-identity $x, y \in G$, $x \neq y$. Then check $\{e, x, y, xy\}$ is a subgroup (closure - write down mult table; inverses - each element is its own inverse). Hence 4 divides $|G|$ by Lagrange.

6. (a) Both $+1$.

(b) $e, (3), (5), (7), (2,2), (2,4), (3,3)$.

(c) Elements of order 2 are those of cycle-shape $(2,2)$. No. of these is $\binom{7}{2} \times \binom{5}{2} \times \frac{1}{2} = 105$.

7. As g has odd order, it is a product of disjoint cycles, all of odd length. These are all even perms., therefore g is even.