## M2PM2 Algebra II    Problem Sheet 8

**1.** (a) For each of the following linear transformations $T : V \to V$, find its eigenvalues, find their algebraic and geometric multiplicities, and determine whether $V$ has a basis consisting of eigenvectors of $T$:

(i) $V = \mathbb{R}^3$, $T(x_1, x_2, x_3) = (-x_1 + x_2 - x_3,\ -4x_2 + 6x_3,\ -3x_2 + 5x_3)$.

(ii) $V$ the vector space of polynomials over $\mathbb{R}$ of degree at most 2, and $T(p(x)) = x(2p(x+1) - p(x) - p(x-1))$ for all $p(x) \in V$.

(b) For which values of $a, b, c$ is the matrix

$$A = \begin{pmatrix} -1 & a & b \\ 0 & 1 & c \\ 0 & 0 & -1 \end{pmatrix}$$

diagonalisable (i.e. $\exists\ P$ such that $P^{-1}AP$ is diagonal) ?

**2.** Prove that if $A$ and $B$ are similar $n \times n$ matrices then they have the same characteristic polynomial.

**3.** Let $A$ be an $n \times n$ matrix and suppose that there exists an invertible $n \times n$ matrix $P$ such that $P^{-1}AP$ is diagonal. If $p(x)$ is the characteristic polynomial of $A$, prove that $p(A) = 0$. (*Don't use the Cayley-Hamilton theorem.*)

**4.** Let $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$, and let $A$ be the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

Prove that the characteristic polynomial of $A$ is $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

**5.** In this question you can use Q4 and Cayley-Hamilton.

(a) Find a $3 \times 3$ matrix which has characteristic polynomial $x^3 - 7x^2 + 2x - 3$.

(b) Find a $3 \times 3$ matrix $A$ such that $A^3 - 2A^2 = I$.

(c) Find a $4 \times 4$ matrix $B$ such that $B^{-1} = B^3 + I$.

(d) Find a real $4 \times 4$ matrix $C$ such that $C^2 + C + I = 0$.

(e) For each $n$ find an $n \times n$ matrix $D$ such that $C^n = I$ but $C \neq I$.

**6.** Let $A$ be an $n \times n$ matrix, and suppose that the only eigenvalue of $A$ in $\mathbb{C}$ is 0. Prove that $A^n = 0$.