M2P2 Algebra II Problem Sheet 7

1. Express $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 3 & 8 & 7 \end{pmatrix}$ as a product of elementary matrices.

2. For $n \times n$ matrices A, B, write $A \sim B$ to mean that B can be obtained from A by a sequence of elementary row operations.

Prove that $A \sim B$ if and only if $B = E_1 \dots E_k A$, where each E_i is an elementary matrix. Deduce that the relation $\sim$ is an equivalence relation.

3. Find the determinants of the following linear transformations T:

(i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ -x_1 - 3x_3,\ x_2 + x_3).$$

(ii) Let V be the vector space of polynomials of degree at most 3 over $\mathbb{R}$, and define $T : V \to V$ by $T(p(x)) = p(1 + x) - p'(1 - x)$ for all $p(x) \in V$.

(iii) Let V be the vector space of all 2×2 matrices over $\mathbb{R}$, let $M = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$, and define $T : V \to V$ by $T(A) = MA$ for all $A \in V$.

4. (i) For T as in part (i) of Q1, calculate the determinant of the linear transformation $q(T)$, where $q(x) = x^3 - 2x^2 + 3x + 1$.

(ii) For T as in part (ii) of Q1, find the eigenvalues of T, and find a basis for each eigenspace. Is there a basis B of V such that the matrix $[T]_B$ is diagonal ?

5. Let $A = \begin{pmatrix} B & C \\ \mathbf{0} & D \end{pmatrix}$, where B is $s \times s$, D is $t \times t$, C is $s \times t$, and $\mathbf{0}$ is the $t \times s$ zero matrix. Prove that $\det(A) = \det(B)\det(D)$.

6. Prove that similarity of $n \times n$ matrices is an equivalence relation.

7. Let E, F be the following bases of $\mathbb{R}^2$: $E = \{(1, 0),\ (0, 1)\}$, $F = \{(1, 3),\ (2, 5)\}$.

(i) Find the change of basis matrix P from E to F.

(ii) Find the change of basis matrix Q from F to E.

(iii) Check that $Q = P^{-1}$.

(iv) Let $v = (a, b) \in \mathbb{R}^2$. Write down the column vectors $[v]_E$ and $[v]_F$, and check that $[v]_E = P[v]_F$.

(v) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (2x_2, 3x_1 - x_2)$. Check that $[T]_F = P^{-1}[T]_E P$.

8. Let V be an n-dimensional vector space over $\mathbb{R}$. Define $GL(V)$ to be the set of all invertible linear transformations from $V \to V$.

(i) Prove that $GL(V)$ is a group under composition.

(ii) Prove that $GL(V) \cong GL(n, \mathbb{R})$. (Recall that $GL(n, \mathbb{R})$ is the group of all invertible $n \times n$ matrices over $\mathbb{R}$, under matrix multiplication.)