M2PM2 Algebra II Problem Sheet 5

1. Find (with proof) all positive integers r such that there is a surjective homomorphism from S_n onto C_r for some n.

2. (a) Show that if G is an abelian group, and N is a subgroup of G, then the factor group G/N is abelian.

(b) Give an example of a non-abelian group G with a normal subgroup N such that both N and G/N are abelian.

(c) Give an example of a group G with subgroups M, N such that $N \lhd G$ and $M \lhd N$, but M is not normal in G.

3. (a) Let G be a cyclic group, and let N be a subgroup of G. Prove that $N \lhd G$ and the factor group G/N is also cyclic.

(b) For each of the following groups G, find all groups H (up to isomorphism) such that there is a surjective homomorphism from G to H:

$$G = D_8, \quad G = D_{12}, \quad G = (\mathbb{Z}, +).$$

4. How many distinguishable necklaces can be made out of 8 beads, 3 of which are coloured yellow, 3 red and 2 blue ?

5. How many distinguishable ways are there of colouring the edges of a square if 6 colours are available, and no colour is used more than once ? How many if colours may be used more than once ?

6. (a) How many distinguishable ways are there of colouring the faces of a regular tetrahedron with 3 colours, if each colour may be used more than once ?

(b) How many distinguishable ways are there of colouring the edges of a regular tetrahedron with 4 colours, if each colour may be used more than once ?

7. Let R be the set of rotations in the symmetry group of a cube.

(a) By writing down as many rotation symmetries as you can think of, show that $|R| \geq 24$.

(b) By counting the possibilities for the 'bottom' face and the 'front' face, show that $|R| \leq 24$.

(c) By considering each rotation as a permutation of the 4 long diagonals of the cube, show that $R \cong S_4$.

8. (a) Dice are made in the usual way by putting the numbers 1,2,3,4,5,6 on the faces of a cube. How many distinguishable dice can be made ?

(b) Dodgy dice are made by putting the numbers 1,2,3,3,4,4 on the faces of a cube. How many distinguishable dodgy dice can be made ?