M2PM2 Algebra II   Problem Sheet 4

1. Which of the following functions ϕ is a homomorphism? For those which are homomorphisms, find Im ϕ and Ker ϕ.

$$\phi : C_{12} \to C_{12} \text{ defined by } \phi(x) = x^3 \ \ \forall x \in C_{12}$$
$$\phi : S_4 \to S_4 \text{ defined by } \phi(x) = x^3 \ \ \forall x \in S_4$$
$$\phi : (\mathbb{Z}, +) \to (\mathbb{Z}_n, +) \text{ defined by } \phi(x) = [x]_n \ \ \forall x \in \mathbb{Z}$$
$$\phi : (\mathbb{R}^*, \times) \to (\mathbb{R}^*, \times) \text{ defined by } \phi(x) = |x| \ \ \forall x \in \mathbb{R}^*$$
$$\phi : (\mathbb{Z}_7, +) \to (\mathbb{Z}_4, +) \text{ defined by } \phi([x]_7) = [x]_4 \ \ \forall [x]_7 \in \mathbb{Z}_7$$

Recall the notation: $[x]_n$ stands for the residue class of x modulo n, and $\mathbb{R}^*$ stands for the set of non-zero real numbers.

2. Let G be a group, and define a function $\phi : G \to G$ by $\phi(g) = g^2$ for all $g \in G$.

(a) Prove that if G is abelian then ϕ is a homomorphism.

(b) Prove that if G is non-abelian then ϕ is not a homomorphism.

3. Let G be a group, and suppose M and N are normal subgroups of G. Show that $M \cap N$ is a normal subgroup of G.

4. Let $G = D_{2n} = \{e, \rho, \ldots, \rho^{n-1}, \sigma, \rho\sigma, \ldots, \rho^{n-1}\sigma\}$, where ρ, σ satisfy the equations $\rho^n = \sigma^2 = e$, $\sigma\rho = \rho^{-1}\sigma$.

(a) Prove that $\sigma\rho^k = \rho^{-k}\sigma$ for all integers k.

(b) Fix a positive integer r. Prove that the cyclic subgroup $\langle \rho^r \rangle$ is a normal subgroup of D_{2n}.

(c) Assume that $n \geq 3$, and let r be a positive integer. Prove that $\langle \rho^r\sigma \rangle$ is not a normal subgroup of D_{2n}.

5. Let p is a prime number greater than 2.

(a) Prove that the dihedral group D_{2p} has exactly three different normal subgroups.

(b) Find all groups H (up to isomorphism) such that there is a surjective homomorphism from D_{2p} onto H.

6. Does there exist a surjective homomorphism

(i) from C_{12} onto C_6 ?

(ii) from C_{12} onto C_5 ?

(iii) from D_8 onto C_4 ?

(iv) from D_8 onto $C_2 \times C_2$?

Give reasons for your answers.

7. Define V to be the following set of permutations in S_4:

$$V = \{e, (12)(34), (13)(24), (14)(23)\}$$

(so V consists of the identity and all permutations of cycle-shape (2,2)).

(i) Show that V is a subgroup of S_4.

(ii) Show that for any $g \in S_4$ and $v \in V - \{e\}$, the element $g^{-1}vg$ has order 2 and is an even permutation. Deduce that $V \triangleleft S_4$.

(iii) The factor group S_4/V has order 6, so by lecs is isomorphic to C_6 or D_6. Which? (Give reasoning.)