**M2PM2 Algebra II** **Problem Sheet 3**

**Assessed work:** *Hand in solutions to starred questions by Monday November 3rd*

**1.** List the cycle-shapes of elements of $A_5$, and calculate how many elements there are of each shape. Check that your answers add up to $|A_5|$.

**2*.** Up to isomorphism, how many different abelian groups are there

(a) of size 21 ?

(b) of size 12 ?

(c) of size 81 ?

Justify your answers.

**3*.** (a) What is the largest order of an element of $S_7$ ?

(b) Show that if $S_7$ has a subgroup which is isomorphic to $D_{2n}$, and $n > 7$, then $n = 10$ or 12.

(c) Does $S_7$ have a subgroup which is isomorphic to $D_{24}$ ? (*Hint: see if you can find elements of $S_7$ satisfying the magic equations for $D_{24}$.*)

**4*.** Use direct products to give examples of groups $G$ with the following properties:

(i) $|G| = 2^n$ and $x^2 = e$ for all $x \in G$, where $n$ is an arbitrary positive integer

(ii) $|G| > 8$, $G$ is non-abelian, and $x^4 = e$ for all $x \in G$

(iii) $G$ is infinite and non-abelian, and $G$ has a subgroup $H$ such that $|G : H| = 2$ and $H$ is abelian (recall $|G : H|$ is the *index* of $H$ in $G$, i.e. the number of distinct right cosets of $H$ in $G$).

**5.** Let $A, B$ be the following matrices over the complex numbers:

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(a) Show that $A^4 = B^4 = I$, $A^2 = B^2$ and $BA = A^3B$.

(b) Deduce that the set $\{A^r B^s : r, s \in \mathbb{Z}\}$ consists of exactly 8 matrices, and write them down.

(c) Let $Q_8$ be the set of matrices in (b). Prove that $Q_8$ is a subgroup of $GL(2, \mathbb{C})$.

(d) Prove that $Q_8 \not\cong D_8$.

**6*.** Let $G$ be a non-abelian group such that $|G| = 8$.

(a) Prove that $G$ has an element $x$ of order 4.

(b) Given that $G$ has an element $y$ such that $y$ has order 2 and $y \neq x^2$, prove that $G \cong D_8$. (*Hint: try to copy the proof in lecs for groups of size 6.*)

**7.** Prove that up to isomorphism, the only non-abelian groups of size 8 are $D_8$ and $Q_8$.

**8.** Let $G$ be a group of size 9. Prove that $G$ must be abelian, in the following steps:

(a) Suppose (for a contradiction) that $G$ is non-abelian. Show that every non-identity element of $G$ has order 3.

(b) Show that there are elements $x, y \in G$ such that $G = \{x^i y^j : 0 \leq i, j \leq 2\}$.

(c) Show that $yx = xy$. (*Hint: if e.g. $yx = xy^2$, consider $(yx)^3$.*)

(d) Obtain a contradiction.

**9.** List all groups of size 9 (up to isomorphism).