M2PM2 Algebra II **Problem Sheet 2**

1. Prove that isomorphism of groups is an equivalence relation (i.e. prove that the relation $\sim$ defined by $G \sim H \Leftrightarrow G \cong H$ is an equivalence relation).

2. Prove that if G, H are groups and $\phi : G \to H$ is an isomorphism, then $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$.

3. Which pairs among the following groups are isomorphic ?

$(\mathbb{Q}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Q}^*, \times)$,
the subgroup $\langle \pi \rangle$ of $(\mathbb{R}^*, \times)$,
the group $(\mathbb{Q} - \{-1\}, *)$, where $a * b = ab + a + b \ \ \forall a, b \in \mathbb{Q} - \{-1\}$

4. (a) Prove that no two of the groups S_5, C_{120} and D_{120} are isomorphic to each other.

(b) Prove that S_3 is isomorphic to D_6.

(c) Prove that $(\mathbb{R}, +)$ is isomorphic to $(\mathbb{R}_{>0}, \times)$, where $\mathbb{R}_{>0}$ is the set of positive real numbers.

(d) Prove that D_8 has two subgroups of size 4 which are not isomorphic to each other.

5. Let G be a group with the property that $x^2 = e$ for all $x \in G$.

(a) Prove that G must be abelian.

(b) Prove that either $|G| \leq 2$, or $|G|$ is divisible by 4.

6. (a) Find the signatures of the following permutations g and h :

$$g = (1278)(359)(46), \quad h = (12)(34)(24895).$$

(b) List the cycle-shapes of elements of the alternating group A_7.

(c) Calculate the number of elements of order 2 in A_7.

7. Let $g \in S_n$. Show that if g has odd order, then g must be an even permutation.