Monday, 19th of May 2008, 14:00-16:00

IMPERIAL COLLEGE LONDON

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Date:	January 18, 2008

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

M2MP1

Real Analysis

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IMPERIAL COLLEGE LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

This paper is also taken for the relevant examination for the Associateship.

M2MP1

Real Analysis

Date: 19th of May, 2008 Time: 14:00 - 16:00

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) Give the definition of $f \to l$ as $x \to a_+$ and $f \to l$, as $x \to a_-$ in terms of ε and δ.

(ii) State the Intermediate Value Property for a continuous function f defined on $[a, b] \subset \mathbb{R}$.

(iii) Prove that if $f : [0, 1] \to [0, 1]$ is continuous, then $\exists\, c \in [0, 1]$, such that $f(c) = c$.

(iv) Let f be continuous on $[0, 2]$. Prove that there are $x, y \in [0, 2]$ such that

$$y - x = 1, \quad f(y) - f(x) = \frac{1}{2}\Big(f(2) - f(0)\Big).$$

Hint: consider $g(x) = f(x + 1) - f(x) - (f(2) - f(0))/2$, $x \in [0, 1]$.

2. (i) State Taylor's Theorem with Lagrange's form of the remainder.

(ii) Find Taylor's formula at 0 with Lagrange's form of the remainder for the function

$$f(x) = x \ln(1 + x^2), \qquad x \in \mathbb{R}.$$

(iii) Prove the inequality

$$(1 + x)^{\frac{1}{n}} > 1 + \frac{x}{n} - \frac{n-1}{2n^2}x^2, \quad x > 0, \quad n = 2, 3, 4, \ldots.$$

3. (i) Define upper and lower Riemann sums for a bounded function
$f : [a, b] \to \mathbb{R}$.

(ii) Prove that if f is continuous on $[a, b]$ then for any $c \in (a, b)$

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

(iii) Show that is f is a non-negative continuous function on $[a, b]$ such that

$$\int_a^b f(x)\, dx = 0,$$

then $f(x) \equiv 0$ on $[a, b]$.

4. (i) Define a directional derivative for a function $f : B_r(a) \to \mathbb{R}$ at $a \in \mathbb{R}^n$, where $B_r(a) = \{x \in \mathbb{R}^n : |x - a| < r\}$.

(ii) Let $f : \mathbb{R}^3 \to \mathbb{R}$, such that $f(x, y, z) = xy + z^2(x + y)$ and let $\nu = (1, 1, 1)$. Find $f'_\nu(3, 2, 1)$.

(iii) Let $f : \mathbb{R}^n \to \mathbb{R}$ and let for any $\alpha > 0$, $x \in \mathbb{R}^n$, we have $f(\alpha x) = \alpha f(x)$. Prove that $f(x) = x \cdot \nabla f(x)$.