**UNIVERSITY OF LONDON**

**IMPERIAL COLLEGE LONDON**

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| Course: | M2P4 |
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| Date: | April 2, 2007 |

**BSc and MSci EXAMINATIONS (MATHEMATICS)**
**MAY–JUNE 2007**

*This paper is also taken for the relevant examination for the Associateship.*

# M2P4 Rings and fields

DATE: examdate TIME: examtime

*Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.*

*Calculators may not be used.*

Setter's signature ..................................................

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**1.** *i)* Give the definitions of a unit, and of an irreducible element of an integral domain.

*ii)* Prove that the ring of polynomials with coefficients in a field is an integral domain.

*iii)* Which of the following polynomials are irreducible over $\mathbb{Z}/3$: (a) $x^2+1$, (b) $x^2+x+1$, (c) $x^3+1$, (d) $x^4+x^2+1$ ? (Justify your answer.)

**2.** *i)* Give the definition of a unique factorization domain (UFD).

*ii)* Explain the key steps of the proof that $\mathbb{Z}[\sqrt{3}]$ is a UFD. (A few sentences will suffice.)

*iii)* Prove that $\mathbb{Z}[\sqrt{-2007}]$ is not a UFD. (Hint: you may want to prove first that 2 is an irreducible element of $\mathbb{Z}[\sqrt{-2007}]$.)

**3.** *i)* Give the definition of a maximal ideal of an integral domain.

*ii)* Prove that in a principal ideal domain every non-zero maximal ideal is generated by an irreducible element.

*iii)* Find all the maximal ideals of $\mathbb{Q}[x]$ containing the polynomial $x^6-1$. (You can use all the results from the course provided you state them clearly.)

**4.** *i)* State and prove Eisenstein's irreducibility criterion.

*ii)* Find all $n \in \mathbb{Z}$ such that $x^3+nx^2+6$ is irreducible over $\mathbb{Q}$.

*iii)* Find the characteristic of the field $\mathbb{Z}[\sqrt{-3}]/(4+\sqrt{-3})\mathbb{Z}[\sqrt{-3}]$.

**5.** *i)* Define what is meant by the degree $[F:K]$ of an extension of fields $K \subset F$. Find $[F:\mathbb{Q}]$ where $F$ is the smallest subfield of $\mathbb{C}$ containing all the roots of $x^6-1=0$.

*ii)* Explain why a regular polygon with 9 sides cannot be constructed using only a ruler and a compass. (A few sentences will suffice; you are not asked to give full details or your argument.)

*iii)* Prove that a regular polygon with 5 sides can be constructed using only a ruler and a compass.