Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M2P4

Rings and Fields

Date: Monday, 22nd May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let R be an integral domain. Define what it means to say that

(i) R is a Euclidean domain;

(ii) R is a unique factorization domain.

What relationship (if any) is there between Euclidean domains and unique factorization domains? (No proof is required.)

Prove that $\mathbb{Z}[\sqrt{2}]$ is an Euclidean domain.

Prove that $\mathbb{Z}[\sqrt{-3}]$ is not a Euclidean domain.

2. Give the definition of an integral domain.

Give the definition of a field.

Prove that every *finite* integral domain is a field.

Let R be an integral domain and let I be an ideal of R. Prove that R/I is a field if and only if I is maximal.

Which of the following are fields? Justify your answers.

(i) $\mathbb{Q}[x]/(x^2+1)\mathbb{Q}[x]$;

(ii) $\mathbb{Z}[x]/(x^2-1)\mathbb{Z}[x]$.

3. Define the characteristic of a field.

Let F be a finite field of characteristic p. Prove the following:

(i) p is a prime number;

(ii) there is an injective homomorphism $\varphi : \mathbb{Z}/p\mathbb{Z} \to F$;

(iii) every subfield in F contains the image of the homomorphism φ as in (ii).

Suppose that $f(x) \in F[x]$, where F is a field, and that the degree of $f(x)$ is 2 or 3. Prove that $f(x)$ is irreducible if and only if $f(x)$ has no roots in F.

Give with justification the numbers of irreducible polynomials in $(\mathbb{Z}/2\mathbb{Z})[x]$

(a) of degree 2;

(b) of degree 3;

(c) of degree 4;

(d) of degree 5.

4. Let $f(x)$ be a polynomial of degree at least 1 and having integer coefficients. State and prove Gauss's Lemma which relates the irreducibility of $f(x)$ as an element of $\mathbb{Z}[x]$ to the irreducibility of $f(x)$ as an element of $\mathbb{Q}[x]$.

   Let p be an odd prime number and let $f_p(x) = 1 + x + x^2 + \ldots + x^{p-1}$. Prove that $f_p(x)$ is irreducible in $\mathbb{Q}[x]$. (You can use results from the course provided that you state them explicitly.)

   Is $f_p(x)$ irreducible in $(\mathbb{Z}/p\mathbb{Z})[x]$?

5. Explain carefully what is meant by saying that a real number is constructible with a ruler and a compass.

   Prove that if a positive number a is constructible then so is $\sqrt{a}$.

   Prove that a regular pentagon is constructible.

   Given an odd prime number p, state (without proof) a necessary and sufficient condition for a regular p-sided polygon to be constructible (that is, constructible using a ruler and a compass).