UNIVERSITY OF LONDON

IMPERIAL COLLEGE LONDON

Course:	M 2 P 4
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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2005

This paper is also taken for the relevant examination for the Associateship.

M 2 P 4 Rings and fields

DATE: examdate TIME: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Setter's signature	....
Checker's signature	....

1. *i)* Give the definition of an integral domain.

ii) Prove that every *finite* integral domain is a field.

iii) Which of the following rings are integral domains? Justify your answers. (You may use any results from the course as long as you clearly state them.)

(a) $\mathbf{Z}[x]/(x^2+1)\mathbf{Z}[x]$;

(b) $\mathbf{Z}[x]/x^2\mathbf{Z}[x]$;

(c) $\mathbf{Z}[x]/(x^2-1)\mathbf{Z}[x]$;

(d) $\mathbf{Z}_2[x]/(x^2+1)\mathbf{Z}_2[x]$;

(e) $\mathbf{Z}_3[x]/(x^2+1)\mathbf{Z}_3[x]$.

2. *i)* Give the definition of a Euclidean domain.

ii) Prove that every Euclidean domain is a principal ideal domain.

iii) Briefly explain why $\mathbf{Q}[x]$ is a Euclidean domain (three or four lines will suffice).

iv) Use Euclid's algorithm to find a generator of the ideal $f(x)\mathbf{Q}[x]+g(x)\mathbf{Q}[x]$ of $\mathbf{Q}[x]$, where $f(x)=x^2-4$ and $g(x)=x^3-2x^2-5x+10$.

3. *i)* Give the definition of the characteristic of a field.

ii) Prove that a finite field of characteristic p, where p is a prime number, has p^n elements for some positive integer n.

iii) Construct a field with 121 elements. (You may use any results from the course as long as you clearly state them.)

iv) Write $x^5+x^3+x^2+1$ as a product of irreducible polynomials in $\mathbf{Z}_2[x]$.

4. *i)* State Gauss's lemma (no proof is required).

ii) Find all integers n for which the polynomial $x^3 + nx + 5$ is reducible in $\mathbf{Q}[x]$.

iii) In which of the following rings is the principal ideal generated by 3 maximal? Justify your answer.

(a) $\mathbf{Z}[\sqrt{-1}]$;

(b) $\mathbf{Z}[\sqrt{-2}]$;

(c) $\mathbf{Z}[\sqrt{-3}]$.

You may use any results from the course as long as you clearly state them.

5. Let $F \subset K$ be fields.

i) Say what it means for an element of K to be algebraic over F.

ii) Let $\alpha \in K$ be algebraic over F. Define the minimal polynomial of α.

iii) Let $F = \mathbf{Q}$, and K be the smallest subfield of $\mathbf{R}$ which contains $\sqrt{2}$ and $\sqrt{5}$. Find the degree of K over F. (Justify your answer.)

iv) Find the minimal polynomial of $\sqrt{2} + \sqrt{5}$ over $\mathbf{Q}$.

You may use any results from the course as long as you clearly state them.