Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2007

This paper is also taken for the relevant examination for the Associateship.

M2P3

Complex Analysis I

Date: Friday, 11th May, 2007 Time: 10 am - 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let $g : \mathbb{C} \to \mathbb{C}$ be a function such that for any pair of points $z_0, z_1 \in \mathbb{C}$ we have

$$|g(z_0) - g(z_1)| \leq \lambda |z_0 - z_1| \tag{1}$$

for some constant λ with $0 < \lambda < 1$.

(a) (i) Define what it means for a function $f : \mathbb{C} \to \mathbb{C}$ to be *continuous* on $\mathbb{C}$.

(ii) Show that any function satisfying Eq. 1 is continuous.

(b) (i) Define what it means for a sequence z_n in $\mathbb{C}$ to be *bounded*.

(ii) Define what it means for a sequence z_n in $\mathbb{C}$ to be *convergent*.

(iii) State (without proof) the *Bolzano-Weierstrass Theorem* in $\mathbb{C}$.

(c) Let $z_0 = 0$ and define a sequence $z_n \in \mathbb{C}$ by $z_{n+1} = g(z_n)$ for $n \in \mathbb{N}$, where g is a function satisfying Eq. 1.

(i) Show that for all $n \in \mathbb{N}$ we have

$$|z_{n+1} - z_n| \leq \lambda^n |z_1 - z_0|.$$

Using the fact that $|z_n - z_0| \leq |z_n - z_{n-1}| + |z_{n-1} - z_{n-2}| + \ldots + |z_1 - z_0|$ deduce that z_n is a bounded sequence.

(ii) Show that there exists a constant $C > 0$ such that if $m > n$ we have

$$|z_n - z_m| \leq \lambda^n C$$

and hence deduce that z_n is a convergent sequence.

(iii) Using (a) above, deduce that there exists a *unique* point $z \in \mathbb{C}$ such that $g(z) = z$.

You may use without proof any results you require about the image of a convergent sequence under a continuous function.

2. (a) Define what it means for a function $f : \mathbb{C} \to \mathbb{C}$ to be *differentiable* at a point $z \in \mathbb{C}$ and define the *derivative* of f at z.

(b) Denote $f(x+iy) = u(x,y) + iv(x,y)$ where $u, v : \mathbb{R}^2 \to \mathbb{C}$. State the *Cauchy-Riemann Equations* for f.

(c) Suppose that f is differentiable at a point $z \in \mathbb{C}$. By taking the limit along the diagonals $z + h$ with $\mathfrak{Re}\, h = \mathfrak{Im}\, h$ and $\mathfrak{Re}\, h = -\mathfrak{Im}\, h$ respectively, show that the derivative of f can be expressed in two different ways as

$$\begin{aligned} f'(z) &= \frac{1}{1+i}(u_x + u_y + iv_x + iv_y) \\ f'(z) &= \frac{1}{1-i}(u_x - u_y + iv_x - iv_y). \end{aligned}$$

(d) Use this to show that the Cauchy-Riemann equations are satisfied for f.

(e) Suppose that $f : \mathbb{C} \to \mathbb{C}$ only depends on the imaginary part of z. Show that if f is differentiable, then it must be constant, stating clearly any general results that you use.

3. (a) State (without proof) the *ML Inequality* for the integral over a smooth contour.

(b) Define the *anti-derivative* of f and state (without proof) the *Fundamental Theorem of Calculus for Contour Integrals*.

(c) Suppose that f is a continuous function and γ a closed smooth contour such that $\int_\gamma f\ dz \neq 0$. Show that for any polynomial p there is at least one point z on γ such that

$$|f(z) - p(z)| \geq \frac{1}{L} \left| \int_\gamma f\ dz \right|$$

where L is the length of γ.

(d) Show that if

$$\int_\gamma f\ dz = \int_\sigma f\ dz$$

whenever γ and σ have the same endpoints, then f has an anti-derivative.

4. (a) (i) Define a *Star-Domain*.

   (ii) State (without proof) *Cauchy's Theorem for a Star-Domain*.

   (iii) State (without proof) *Cauchy's Integral Formula* for an analytic function $f : D_R(z_0) \to \mathbb{C}$.

   (b) Let $f : \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = \frac{\overline{z}}{z - a}.$$

   and γ be the circle $C_r(0)$ of radius $r > 0$ and centre 0.

   (i) If $a = 0$ show by direct evaluation that

$$\int_\gamma f \; dz = 0. \tag{2}$$

   (ii) If $a \neq 0$ use the fact that $z\overline{z} = |z|^2$ to decompose f as $f(z) = g(z)h(|z|)$ where

$$g(z) = \frac{A}{z - a} + \frac{B}{z}.$$

   for appropriate constants A and B. Use Cauchy's Integral Formula to show that Eq. 2 also holds if $0 < |a| < r$.

   (iii) Use Cauchy's Theorem to evaluate $\int_\gamma f \; dz$ when $|a| > r$.

5. (a) State (without proof) *Taylor's Theorem* for an analytic function $f : D_R(z_0) \to \mathbb{C}$. You should include an integral expression for $f^{(n)}(z_0)$, the n^{th} derivative of f at z_0.

   (b) Deduce *Cauchy's Estimate*

$$|f^{(n)}(z_0)| \leq \frac{n! \, M(r)}{r^n}$$

   for any $0 < r < R$, where $M(r)$ is an upper bound on $|f(z)|$ on the circle $C_r(z_0)$, so that $|f(z)| \leq M(r)$ for all $z \in C_r(z_0)$.

   (c) Suppose that f is an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that for some constant $K > 0$

$$|f(z)| \geq K$$

   for all $z \in \mathbb{C}$. By considering $g(z) = 1/f(z)$ show that f is constant on the whole of $\mathbb{C}$.

   (d) Suppose that f is an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that for some $n \in N$, Cauchy's Estimate is an equality for all $r > 0$. Show that for any $m \in N$

$$|f^{(m)}(z_0)| \leq |f^{(n)}(z_0)| \, \frac{m!}{n!} \, r^{n-m}.$$

   Deduce that $f(z) = c(z - z_0)^n$ for some constant $c \in \mathbb{C}$.