Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M2P3

Complex Analysis I

Date: Thursday, 11th May, 2006 Time: 10 am - 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) (i) State (without proof) the Triangle Inequality in $\mathbb{C}$ and draw a diagram to indicate its geometric significance.

(ii) Show that for any four points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ we have

$$|z_1 - z_4| \leq |z_1 - z_2| + |z_2 - z_3| + |z_3 - z_4|$$

(b) Let $z_n \in \mathbb{C}$ be a sequence in $\mathbb{C}$. Define what it means for z_n to be

(i) *Convergent*.

(ii) *Cauchy*.

(iii) *Bounded*.

(c) Prove that

(i) Every Cauchy sequence $z_n \in \mathbb{C}$ is bounded.

(ii) Every convergent sequence $z_n \in \mathbb{C}$ is Cauchy.

(d) (i) Suppose that $z_n \in \mathbb{C}$ is a Cauchy sequence. Show that any two convergent subsequences of z_n must have the same limit. *Hint:* use (a)(ii) above.

(ii) Give an example of a bounded sequence $w_n \in \mathbb{C}$ which has two convergent subsequences with different limits. Deduce that such a sequence w_n cannot be convergent.

2. (a) Define what it means for a function $f : \mathbb{C} \to \mathbb{C}$ to be *differentiable* at a point $a \in \mathbb{C}$ and define the *derivative* of f at a.

(b) Denote $f(x+iy) = u(x,y) + iv(x,y)$ where $u, v : \mathbb{R}^2 \to \mathbb{C}$. State the *Cauchy-Riemann Equations* for f and prove that they are satisfied if f is differentiable. In such a case, express the derivative of f in *two* different ways in terms of the partial derivatives of u and v.

(c) Verify directly that the Cauchy-Riemann Equations are satisfied in the whole of $\mathbb{C}$ for

$$f(z) = \exp z$$

(d) Let $f : \mathbb{C} \to \mathbb{C}$ be differentiable on the whole of $\mathbb{C}$. Suppose that for some constant $c \in \mathbb{R}$ we have $u(x,y) = c\, v(x,y)$ for all $(x+iy) \in \mathbb{C}$. Prove that f is constant, stating clearly any general results that you use and paying careful attention to the special case $c = 0$.

3. (a) State (without proof) the *ML Inequality* for the integral over a smooth contour.

(b) (i) Let f be an analytic function $f : D_R(z) \to \mathbb{C}$, where $D_R(z)$ is an open disc of radius R centred at z. Let $[z, z+h]$ be the straight segment joining z to $z+h$. Given $\epsilon > 0$ show that there exists a $\delta > 0$ such that if $|h| < \delta$ then

$$\left| \int_{[z,z+h]} f(w) - f(z) \; dw \right| \leq \epsilon \, |h|$$

(ii) Deduce that

$$\lim_{h \to 0} \left(\frac{1}{h} \int_{[z,z+h]} f(w) \; dw \right) = f(z)$$

(c) (i) State (without proof) *Cauchy's Theorem for a Triangle*.

(ii) Define a *Star-Domain*.

(iii) State *Cauchy's Theorem for a Star-Domain* and show how it can be deduced from (b)(ii), (c)(i) and the Fundamental Theorem of Calculus (which may be used without proof).

4. (a) Let g be a continuous function $g : [a, b] \to \mathbb{C}$ where $[a, b]$ is an interval on the real line. Show that

$$\left| \int_a^b g(t) \; dt \right| \leq \int_a^b |g(t)| \; dt$$

Hint: first consider the case where $\int_a^b g(t) \; dt$ *is real.*

(b) State (without proof) *Cauchy's Integral Formula* for an analytic function $f : \mathbb{C} \to \mathbb{C}$.

(c) Deduce that for any $a \in \mathbb{C}$ and any $r > 0$, we have

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) \; dt.$$

(d) Suppose that $a \in \mathbb{C}$ is a point such that locally f has maximum modulus at a, so that for some $R > 0$ we have $|f(a)| \geq |f(z)|$ for all $z \in D_R(a)$.

(i) Suppose that for some $z \in D_R(a)$ we have $|f(z)| < |f(a)|$. Show that there exists an $\epsilon > 0$ such that $|f(w)| < |f(a)| - \epsilon$ for all w in a neighbourhood of z.

(ii) Let $r = |z - a|$. By subdividing the circle $D_r(a)$ into two arcs, on one of which $|f(w)| \leq |f(a)| - \epsilon$ show that

$$\int_0^{2\pi} |f(a + re^{it})| \; dt < 2\pi |f(a)|.$$

(iii) Using (a) and (c) above obtain a contradiction and hence deduce that in fact we must have $|f(a)| = |f(z)|$ for all $z \in D_R(a)$.

5. (a) State (without proof) *Taylor's Theorem* for an analytic function $f : D_R(z_0) \to \mathbb{C}$. You should include an integral expression for $f^{(n)}(z_0)$, the n^{th} derivative of f at z_0.

(b) Deduce *Cauchy's Estimate*

$$|f^{(n)}(z_0)| \leq \frac{n!\, M(r)}{r^n}$$

for any $0 < r < R$, where $M(r)$ is an upper bound on $|f(z)|$ on the circle $C_r(z_0)$, so that $|f(z)| \leq M(r)$ for all $z \in C_r(z_0)$.

(c) State and prove *Liouville's Theorem* for bounded analytic functions $f : \mathbb{C} \to \mathbb{C}$.

(d) Suppose that f is an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that for some $n \in \mathbb{N}$ there is a constant $K > 0$ such that

$$|f(z)| \leq K|z|^n$$

for all $z \in \mathbb{C}$. Show that f is a polynomial of degree at most n.