1. Let $f = u + iv$ be a function of a complex variable $z = x + iy$ defined for all $z \in \mathbb{C}$.

(a) (i) Let $a \in \mathbb{C}$. Define what it means for f to be $\mathbb{C}$-*differentiable* at a. Show that if f is $\mathbb{C}$-differentiable at a then f is continuous at a.

(ii) Deduce that if f is holomorphic on $\mathbb{C}$ then $\{z \in \mathbb{C} : f(z) \neq 0\}$ is an *open* subset of $\mathbb{C}$.

(b) (i) Write down the Cauchy-Riemann equations that relate the partial derivatives $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, $v_x = \partial v/\partial x$, $v_y = \partial v/\partial y$, and verify that they are equivalent to the matrix equation

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Find the matrix of partial derivatives in the special case in which $f(z) = iz$.

(ii) Determine a real-valued function $v = v(x, y)$ of x, y such that $f(z) = xy + iv$ is holomorphic on $\mathbb{C}$.

2. (a) Give the definition of the principal value logarithm of z, denoted $\log z$, which is holomorphic on the cut plane $\mathbb{C} \setminus (-\infty, 0]$ [a proof of this is not required].

(b) Let $\beta\colon [0,1] \to \mathbb{C}$ be the curve given by $\beta(t) = t - t^2 + it$. Use an appropriate version of the fundamental theorem of calculus to compute the contour integral $\displaystyle\int_\beta \frac{1}{z+1}\,dz$

$\left[\text{You may assume that the complex derivative of } \log z \text{ on } \mathbb{C} \setminus (-\infty, 0] \text{ is } \frac{1}{z}.\right]$

(c) Let $\gamma\colon [0, \frac{\pi}{4}] \to \mathbb{C}$ be the curve $\gamma(\theta) = 2e^{i\theta}$. Use the ML inequality to prove that

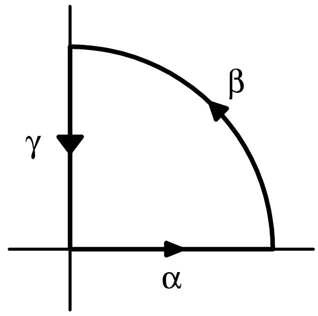
$$\left| \int_\gamma \frac{1}{z^4+1}\,dz \right| \leqslant \frac{\pi}{30}.$$

3. (a) Let g, h be holomorphic functions defined on an open set Ω of $\mathbb{C}$. Suppose that $a \in \Omega$, $h(a) = 0$ and $h'(a) \neq 0$, so that a is a simple pole of the function $f = \dfrac{g}{h}$ [this you may assume]. Define the residue of f at a, denoted $\mathrm{Res}_a f$, and show that $\mathrm{Res}_a f = \dfrac{g(a)}{h'(a)}$.

(b) Let $\sigma = e^{i\pi/4}$. Express the roots of z^4+1 in terms of σ, and show that $f(z) = \dfrac{1}{z^4+1}$ has a simple pole at $z = \sigma$. Deduce that $\mathrm{Res}_\sigma f = -\frac{1}{4}\sigma$.

(c) Compute the residue of the function $F(z) = \dfrac{1}{z \sin z}$ at both $z = \pi$ and $z = 0$.

4. The closed contour $\Gamma = \alpha + \beta + \gamma$ illustrated is the join of a straight line segment along the positive real axis of length R, a quarter circle of radius R and a straight line segment back to 0 along the imaginary axis. Let



$$f(z) = \frac{1}{z^4+1}.$$

(a) Write down a parametrization of the curve β, and that of the segment γ.

(b) Show that $\int_\beta f(z)dz \to 0$ as $R \to \infty$.

(c) Show that $\int_\gamma f(z)dz = -i \int_\alpha f(z)dz$.

(d) Use the residue formula to show that

$$\int_0^\infty \frac{1}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}.$$

[You may assume the result of Question 3(b).]

5. (a) Let $z = e^{i\theta}$ be a complex number of modulus one. Show that $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, and state a similar expression for $\sin\theta$.

(b) Let n be a positive integer. Show that, when $\cos^{2n}\theta$ is expanded in powers of z and $\frac{1}{z}$, its constant term equals $2^{-2n}\binom{2n}{n}$ $\left[\text{where } \binom{2n}{n} \text{ is the binomial coefficient } \frac{(2n)!}{n!\,n!}\right]$. Show that the constant term in $\sin^{2n}\theta$ is the same number.

(c) Use contour integration to evaluate the integral

$$\int_0^{2\pi} (\sin^{2n}\theta + \cos^{2n}\theta)\,d\theta.$$

Verify that your answer is correct for $n=1$.