Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

MAY-JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M2P3 Complex Analysis I

Date: Wednesday, 12 May 2004 Time: $14:00 - 16:00$

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Give the definitions of a smooth path and a contour. Give the definition of the integral

$$\int_\gamma f(z)\,dz,$$

when γ is a smooth path and when γ is a contour.

(b) State and prove the ML-inequality including the supporting lemma.

(c) Let γ be the following path

$$\gamma(t) = Re^{it}, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2},$$

where $R > 1$. Prove that

$$\left| \int_\gamma \frac{\log z}{z} dz \right| \le \pi (\ln R + 2).$$

where $\log z$ is the principal value logarithm.

2. (a) State and prove the fundamental theorem of calculus for contour integration.

(b) Prove that if a continuous function f has an antiderivative in an open set Ω then $\int_\gamma f(z)\,dz = 0$ for any closed contour γ in Ω. Hence, show that the function $f(z) = \frac{e^z}{z}$ has no antiderivative in $\mathbb{C} \setminus \{0\}$.

(c) Prove that if f is an analytic function in $\mathbb{C}$ then the function $g(z) = f\left(\frac{1}{z^2}\right)$ has an antiderivative in $\mathbb{C} \setminus \{0\}$.

3. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{1}$$

(a) Prove that the series (1) converges for all $z \in \Omega$ where

$$\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 1\}.$$

(b) Define what it means that a series converges uniformly. Prove that the series (1) converges uniformly in the set

$$\Omega_\varepsilon = \{z \in \mathbb{C} : \operatorname{Re} z > 1 + \varepsilon\},$$

for any $\varepsilon > 0$.

(c) Prove that the sum of the series (1) is an analytic function in Ω. State clearly all the results used.

4. (a) State the residue theorem (no proof is required).

   (b) Evaluate the integral

$$\int_\gamma \frac{z^3 e^{iz}}{\left(z^2+1\right)^2} dz$$

   where $\gamma = [-R, R] + C_+(0, R)$ is the semicircular contour and $R > 1$. State clearly all the results used.

   (c) Using part (b) and the Jordan lemma, evaluate the following integral:

$$\int_{-\infty}^{+\infty} \frac{t^3 \sin t}{\left(t^2+1\right)^2} dt.$$

5. (a) Prove that any bounded analytic function on $\mathbb{C}$ is constant. State clearly all the results used in the proof.

   (b) Let f be an analytic function in $\mathbb{C} \setminus \{a, b\}$, where a, b are two distinct points in $\mathbb{C}$. Let f have a pole at each point a,b and let $f(z) \to 0$ as $z \to \infty$. Prove that the function f can be represented in the form

$$f(z) = \sum_{k=1}^{n} \frac{\alpha_k}{(z-a)^k} + \sum_{l=1}^{m} \frac{\beta_l}{(z-b)^l},$$

   for some positive integers n, m and complex numbers α_k, β_l.