Complex Analysis (M2P3)
Exam paper 2002/2003

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Q 1 (a) Give the definition of $\mathbb{C}$-differentiability of a function $f(z)$. Prove that the function $f(z) = (\operatorname{Re} z)^2 + i|z|^2$ is $\mathbb{C}$-differentiable at $z = 0$.

(b) State the necessary and (separately) sufficient conditions for $\mathbb{C}$-differentiability involving the Cauchy-Riemann equations.

(c) Prove that the above function $f(z)$ is not $\mathbb{C}$-differentiable at any point $z \neq 0$.

(d) Let $u(x, y) = \cosh x \cos y$. Find a function v such that u and v satisfy the Cauchy-Riemann equations at all points. Hence, prove that the function $f = u + iv$ is analytic in $\mathbb{C}$.

Q 2 (a) State and prove the Cauchy integral formula (state clearly all results used).

(b) Using the Cauchy integral formula, evaluate the integral

$$\int_{C(0,1)} \frac{\cos z}{z} dz.$$

Hence, prove that

$$\int_0^{2\pi} \cos(\cos\theta) \cosh(\sin\theta)\, d\theta = 2\pi.$$

Q 3 Let f be a function which is analytic in the punctured disk $D'(0, R)$, $R > 0$.

(a) State Laurent's theorem about a Laurent series expansion of f at 0.

(b) Let $\{c_n\}_{n \in \mathbb{Z}}$ be the coefficients in the Laurent series expansion of the function f at 0. Prove that, if $|f(z)| \leq M$ for all z such that $|z| = r$ (where $0 < r < R$ and M is a constant), then

$$|c_n| \leq M r^{-n}.$$

(c) Assume that in addition the function f is bounded in $D'(0, R)$. Prove that 0 is a removable singularity for f.

Q 4 (a) Give the definition of a residue and state the residue theorem.

(b) For the function

$$f(z) = \frac{z e^{iz}}{z^2 + 1}$$

find all isolated singularities and evaluate the residue at each singularity.

(c) By firstly evaluating the contour integral $\int_\gamma f(z)\, dz$, where γ is a suitable closed semicircular contour and $f(z)$ is the above function, prove that

$$\int_{-\infty}^{+\infty} \frac{t \sin t}{t^2 + 1} dt = \frac{\pi}{e}.$$

State clearly every result used.

Q 5 (a) State the necessary and sufficient condition for a continuous function to have an antiderivative in a domain.

(b) Let a function $f(z)$ be analytic in $\Omega := \mathbb{C} \setminus \{0\}$. Prove that f has an antiderivative in Ω if and only if $\mathrm{res}_0 f = 0$.

Hint: Use part (a) and the Laurent series expansion of f at 0.

(c) Decide (and prove) which of the following functions

$$\text{(i)} \quad \exp\left(\frac{1}{z}\right) \qquad \qquad \text{(ii)} \quad \frac{\sinh z}{z^3}$$

has an antiderivative in $\mathbb{C} \setminus \{0\}$.