M2P2 ALGEBRA II

1. (a) If G and H are groups, define what is meant by the statement that G is isomorphic to H.

(b) Give brief definitions of the following:

an abelian group
the cyclic group C_n
the alternating group A_n

(c) Define the dihedral group D_{2n}. Show that it contains two elements ρ and σ such that

$$\rho^n = e, \quad \sigma^2 = e, \quad \text{and } \sigma\rho = \rho^{-1}\sigma.$$

(d) How many different abelian groups of size 12 are there, up to isomorphism? Justify your answer, stating any standard results you use.

(e) Prove that D_{12} is not isomorphic to A_4.

(f) Prove that D_{12} is isomorphic to $D_6 \times C_2$.

2. (a) Let G and H be groups. Define what is meant by

a homomorphism $\phi : G \to H$
the kernel of ϕ, $\ker \phi$
the image of ϕ, $\operatorname{Im} \phi$
a normal subgroup N of G
the factor group G/N.

(b) Prove that $\ker \phi$ is a subgroup of G. Prove further that it is a normal subgroup.

(c) State a result which links the factor group $G/\ker \phi$ with the image of ϕ.

(d) Let p be an odd prime, and let D_{2p} be the dihedral group of size $2p$. Find all the normal subgroups of D_{2p}. Justify your answer.

(e) Find all groups H, up to isomorphism, such that there is a surjective homomorphism from D_{2p} onto H.

3. (a) Calculate the number of distinguishable necklaces that can be made from 7 beads, three of which are red, two of which are yellow, and two of which are blue.

(b) Let V be the vector space over $\mathbb{R}$ consisting of all polynomials in x of degree at most 3. Let $T : V \to V$ be the linear transformation defined by

$$T(p(x)) = p(x+1) \,+\, p(x-1)$$

for all polynomials $p(x) \in V$. Calculate the determinant of T.

(c) Express the matrix

$$\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$$

as a product of elementary matrices.

4. (a) Define the characteristic polynomial of an $n \times n$ matrix.

(b) State the Cayley-Hamilton theorem for $n \times n$ matrices.

(c) Let $n > 1$, let $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$, and let A be the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ & & & & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{pmatrix}$$

Prove that the characteristic polynomial of A is

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

(d) Find a real 3×3 matrix A such that $A^3 = I$ but $A \neq I$.

(e) Find a real 3×3 matrix B such that $B^{-1} = B^2 - I$.

(f) Find a real 4×4 matrix C such that $C^2 = C + I$.

5. (a) Define what is meant by an inner product space over $\mathbb{R}$.

(b) Let V be an inner product space, and let $v_1, \ldots, v_r$ be vectors in V. What is meant by the statement that the set $\{v_1, \ldots, v_r\}$ is orthonormal ?

Prove that if $\{v_1, \ldots, v_r\}$ is an orthonormal set, then it is linearly independent.

(c) Let V be an inner product space of dimension n, and suppose that $\{v_1, \ldots, v_{n-1}\}$ is an orthonormal set of vectors in V. Prove that V contains a vector v_n such that $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.

(d) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Prove that

$$(x, y) = x^T A y \qquad (x, y \in \mathbb{R}^2)$$

defines an inner product on $\mathbb{R}^2$. Find an orthonormal basis of $\mathbb{R}^2$ with respect to this inner product.