UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc EXAMINATION (MATHEMATICS) MAY – JUNE 2005
This paper is also taken for the relevant examination for the Associateship

M2P2 Groups, Rings and Numbers

DATE: ????? 2005 TIME: ?????

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Prove that a group in which the square of every element is the identity is an abelian group.

Let $(G, *)$ be a group (where G is a set and $*$ is a binary operation on G). For $g \in G$ let g^{-1} denote the inverse of g in that group, and let e be the identity element. Define a binary operation $\bullet$ on G by the following rule: $g \bullet h = g * h^{-1}$. Prove that if $(G, \bullet)$ is a group then $x \bullet x = e$ for every $x \in G$ and hence $(G, \bullet)$ is abelian. Deduce the conditions on $(G, *)$ which are necessary and sufficient for $(G, \bullet)$ to be a group.

Let $(G, *)$ be a group and $\varphi : G \to G$ be a mapping such that $\varphi(g) = g^{-1}$. Prove that φ is an isomorphism of $(G, *)$ onto itself if and only if $(G, *)$ is abelian.

2. Let $(G, *)$ be a finite group of order n and let $g \in G$. Prove that the order of g divides n (if you are using Lagrange's theorem you must prove it).

Let p be a prime number, let $\mathbf{Z}_p = \{[a]_p \mid a \in \mathbf{Z}\}$ be the set of the residues of the integers modulo p, let $*$ be the multiplication of the elements of $\mathbf{Z}_p$ defined by $[a]_p * [b]_p = [ab]_p$. Prove that $(\mathbf{Z}_p \setminus [0]_p, *)$ is a group.

Find all the positive integers less than 60 which divide $5^{11} - 1$.

3. Let $\Omega = \{1, 2, ..., n\}$. Define the symmetric group S_n of Ω. Prove that every element of S_n can be written as a product of disjoint cycles. Define the alternating group A_n of Ω.

Let Δ be a regular n-gon having Ω as the set of vertices and whose edges are the pairs $\{i, i+1\}$ for $1 \leq i \leq n$ (the addition is modulo n). Let D be the symmetry group of Δ, considered as a permutation group of Ω. Let $t = (1, 2, ..., n)$, $a = (1)(2, n-1)(3, n-2)...$ be elements of D. Prove that every element $d \in D$ can be written in the form $d = t^m a^\varepsilon$, where $0 \leq m \leq n-1$, $\varepsilon \in \{0, 1\}$. Prove that d is of order 2 whenever $\varepsilon = 1$.

Deduce necessary and sufficient conditions on n for D to be a subgroup of A_n.

Turn over...

4. Let $\varphi : G \to H$ be a homomorphism. Define the kernel $\ker(\varphi)$ and the image $\mathrm{Im}(\varphi)$ of φ.

Prove that $\mathrm{Im}(\varphi)$ is a subgroup of H and that $\ker(\varphi)$ is a *normal* subgroup of G. Give an example of a homomorphism when $\mathrm{Im}(\varphi)$ is *not* normal in H.

Construct a surjective homomorphism φ of the symmetric group S_4 of degree 4 on the symmetric group S_3 of degree 3. What is the kernel of φ? Justify your construction and the answer.

5. Let C be 3-dimensional cube with vertices $(\pm 1, \pm 1, \pm 1)$ in $\mathbf{R}^3$. Let A be the symmetry group of C.

Considering the (longest) diagonals of C construct a homomorphism φ of G onto the symmetric group S_4 of degree 4.

Prove that the homomorphism φ is surjective and that its kernel is of order 2 generated by the 'central symmetry' $\tau : x \mapsto -x$, $x \in \mathbf{R}^3$.

Thus calculate the order of G.

Prove that the subgroup of G consisting of rotations of C is isomorphic to S_4.

